Theorems for the orbits of a photon near a black hole

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Recalling that $u = \dfrac{1}{r}$ and $r > 2GM$ (outside the horizon) so that $u < \dfrac{1}{2GM}$ and $1 - 2GMu > 0$, the following takes place.

THEOREM 2: Given that the parameter b is defined only outside the horizon so that $1 - 2GMu > 0$, and its formula is

$$b = \frac{r \sin\alpha}{\sqrt{1 - \dfrac{2GM}{r}}} = \frac{\sin\alpha}{u\sqrt{1 - 2GMu}}, \quad \text{and} \quad \frac{1}{b^2} = \frac{u^2|1 - 2GMu|}{\sin^2\alpha},$$

the under-the-root expression $f(u) = \dfrac{1}{b^2} + 2GMu^3 - u^2$ in formula (5)

$$\frac{du}{d\varphi} = u' = \pm\sqrt{\frac{1}{b^2} + 2GMu^3 - u^2} \tag{(5)}$$

and the under-the-root expression $h(r) = 1 - \dfrac{b^2}{r^2}\left(1 - \dfrac{2GM}{r}\right)$ in formula (2)

$$\begin{aligned} \frac{dr}{dt} &= \pm\left(1 - \frac{2GM}{r}\right)\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)} \qquad ((2)) \\ \frac{d\varphi}{dt} &= \frac{b}{r^2}\left(1 - \frac{2GM}{r}\right) \end{aligned}$$

are always non-negative

PROOF: Consider

$$f(u) = \frac{1}{b^2} + 2GMu^3 - u^2 = \frac{1}{b^2} - u^2(1 - 2GMu)$$

and apply the formula for $\dfrac{1}{b^2}$:

$$\begin{aligned} f(u) &= \frac{u^2|1 - 2GMu|}{\sin^2\alpha} - u^2(1 - 2GMu) \geq \frac{u^2|1 - 2GMu|}{\sin^2\alpha} - u^2|1 - 2GMu| \\ &= u^2|1 - 2GMu|\left(\frac{1}{\sin^2\alpha} - 1\right) \geq 0 \end{aligned}$$

- because all the factors are non-negative.

Now consider

$$h(r) = 1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right).$$

According to the formulas for b,

$$\frac{b^2}{r^2}\left(1-\frac{2GM}{r}\right)=\sin^2\alpha\le 1$$

so that $h(r)\ge 0$ too.

The equality is reached when...

- $\sin^2\alpha = 1$, i.e. when the velocity vector of a photon is perpendicular to the radius $r(\varphi)$, which happens only when the photon is outside the circular orbit reaching the closest distance to the center. Or...
- $u=0$ which happens at infinity, or...
- $u=\frac{1}{2GM}$, i.e. at the horizon.■

LEMMA: The function $g(u) = 2GMu^3 - u^2 \le 0$ (while $u \in \left[0;\ \frac{1}{2GM}\right]$) reaches its minimum $-\frac{1}{27(GM)^2}$ at $u=\frac{1}{3GM}<\frac{1}{2GM}$, or $r=3GM$ (i.e. on the circular orbit).

PROOF: The derivative $g'(u)=6GMu^2-2u$, $g'(u)=0$ for $u_{circ}=\frac{1}{3GM}$ so that $g(u_{circ})=-\frac{1}{27(GM)^2}=-\frac{1}{b_{crit}^2}$. ■

THEOREM 3: The shape of the trajectory $r(t)$ defined by the ODEs (2) coincides with the shape of the curve $r(\varphi)$ defined by the ODE (2) for $u=\frac{1}{r}$.

PROOF: Consider

$$\frac{du}{d\varphi}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial\varphi}=-\frac{1}{r^2}\frac{\partial r}{\partial\varphi}.$$

We obtain $\frac{\partial r}{\partial\varphi}$ dividing the ODEs (2):

$$\begin{aligned}\frac{du}{d\varphi} &= -\frac{1}{r^2}\frac{\partial r}{\partial\varphi}=\pm\frac{1}{b}\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2GM}{r}\right)}\\ &= \pm\sqrt{\frac{1}{b^2}-\frac{1}{r^2}\left(1-\frac{2GM}{r}\right)}=\pm\sqrt{\frac{1}{b^2}-u^2\left(1-2GMu\right)}.\end{aligned}$$

i.e. the ODE (5). ■

Inbound path $r(\varphi)$: from infinity (area A) toward the black hole

Motion of a photon from infinity toward the black hole means that $u = \frac{1}{r}$ increases from zero...

1. Either reaching its maximum $u_{\max}$ outside the circular orbit in area A (bypassing the black hole) when $b > b_{crit} = 3GM\sqrt{3}$;
2. Or crossing the circular orbit into area B and then area C asymptotically approaching the center as a finite spiral when $b < b_{crit}$.
3. Or asymptotically approaching the circular orbit and winding around it as an infinite spiral when $b = b_{crit}$;

In order to specify the inbound motion, we must set the initial value for u' in formula (5) choosing the "+" sign.

Case 1 *Bypassing a black hole:* $\boxed{b > b_{crit} = 3GM\sqrt{3}.}$ *As* u *grows from* 0 *while the function* $g(0) = 0$ *and decreases,* $f(u) = \frac{1}{b^2} + g(u)$ *decreases from* $\frac{1}{b^2}$ *until it reaches* $f_{\min} = f(u_{circ}) \geq 0$ *by the Lemma. We are to prove that* $f_{\min} = 0$ *reaching zero at* $u_{\max} < u_{circ}$ *(*u_{circ} *delivers minimum to* $g(u)$ *in the Lemma).*

$$g(u) > -\frac{1}{b_{crit}^2},$$

then

$$f(u) = \frac{1}{b^2} + g(u) > \frac{1}{b^2} - \frac{1}{b_{crit}^2}.$$

However,

$$\frac{1}{b^2} - \frac{1}{b_{crit}^2} < 0$$

strictly at $u = u_{circ}$*. As* $f(u)$ *cannot be negative by Theorem 2, after reaching* 0 *at some* $u_{\max} < u_{circ}$*,* $f(u)$ *must increase. Therefore,* $u_{\max}$ *is the closest approach of the photon,* $u_{\max} < u_{circ}$*. After reaching* $u_{\max}$*,* u *decreases because the sign at the root must be changed from "+" to "-" in the ODE (5)* $u' = \pm\sqrt{f(u)}$*;* u *decreases from* $u_{\max}$ *to* 0 *while* $f(u)$ *increases along the same graph backward.*

Case 2 *Capture by a black hole:* $\boxed{b < b_{crit} = 3GM\sqrt{3}.}$ *Following the steps of Case 1, now, however, we come to the opposite inequality*

$$\frac{1}{b^2} - \frac{1}{b_{crit}^2} > 0$$

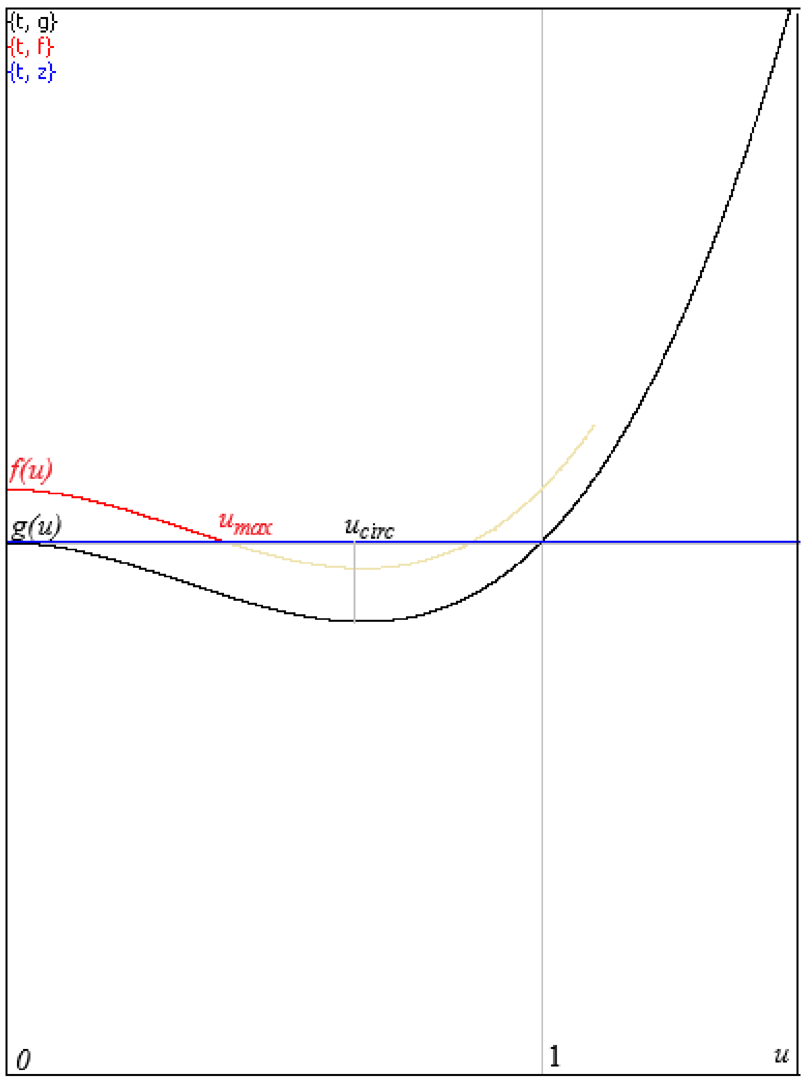

Figure 1: Case 1. Bypassing a black hole: $b > b_{crit}$

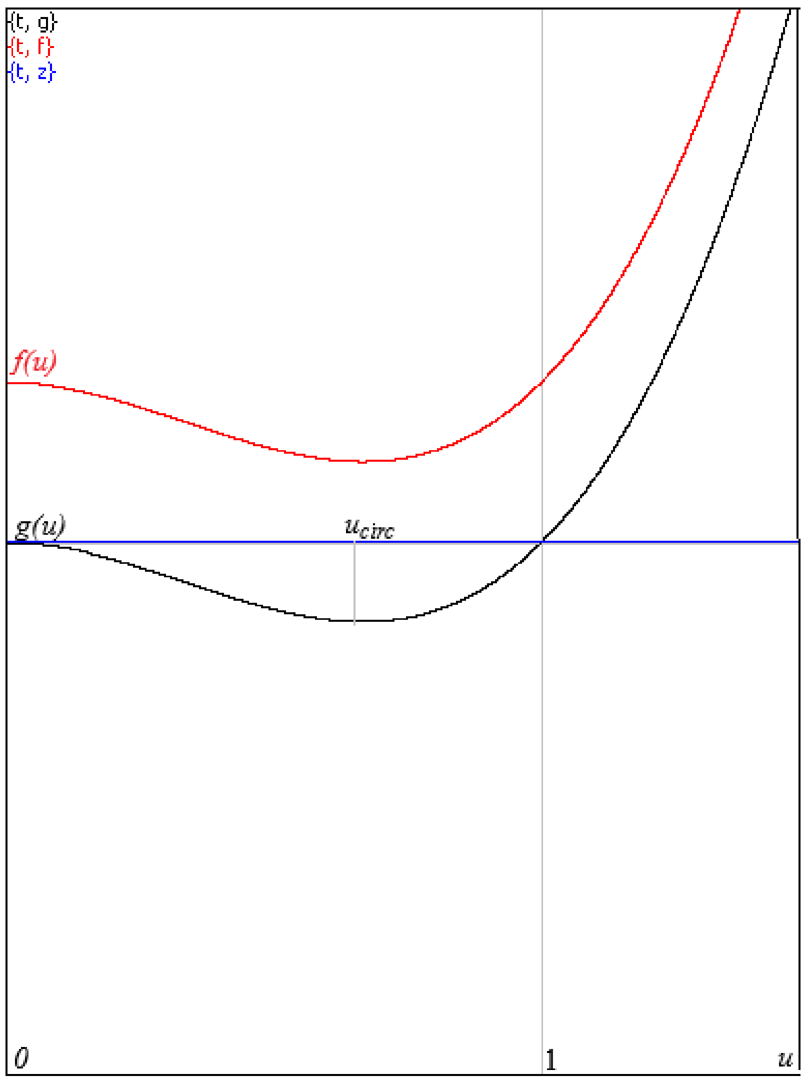

Figure 2: Case 2. Capture by a black hole: $b < b_{crit}$

meaning that $f(u)$ never reaches zero. With the sign "+" in (5), u keeps monotonously increasing so that the photon moving along a spiral crosses the circular orbit and then also the horizon.

Case 3 *Capture by the circular orbit:* $\boxed{b = b_{crit} = 3GM\sqrt{3}.}$ *Unlike the Cases 1, 2, now*

$$\frac{1}{b^2} - \frac{1}{b_{crit}^2} = 0.$$

Again, $f(u) = \frac{1}{b^2} + g(u)$ *keeps decreasing from* $\frac{1}{b^2}$ *towards* 0*, but it never reaches it, as it follows from the next ...*

THEOREM 3: If $b = b_{crit}$ and a photon moves from infinity toward a black hole, u increases and $f(u)$ decreases but never reaches 0, meaning that the photon approaches the circular orbit along an infinite spiral but never crosses or touches the circular orbit.

PROOF: Suppose the opposite, i.e. that $f(u)$ does reach zero at some point $(\varphi,\ r_{circ})$ on the circumference of the circular orbit. Then , at this point φ, we have that $u' = 0$ by formula (5), and $u = \frac{1}{r_{circ}}$. However, according to Theorem 1, there is another solution for these initial values: the exact circular solution. Both these solutions satisfy a regular ODE (6)

$$u'' = 3GMu^2 - u \tag{(6)}$$

which cannot have two different solutions for the same initial values. This controversy proves the Theorem. ■

CONCLUSION: The case of $b = b_{crit}$ is remarkable in that the incoming photon is caught by the circular orbit rather than by the horizon, winding around the circumference in a tight spiral. As we will see further, outgoing photons moving from a vicinity near the horizon away having the parameter $b = b_{crit}$, are also caught into an infinite spiral but approach the circular orbit from inside.

Inbound or outbound path $r(\varphi)$ in area B: capture or escape

When in area B, there are the following possibilities of a photon.

1. If it was emitted so that $u = \frac{1}{r}$ increases (the "+" sign in ODE (5)) and the motion is inbound with any b, u will increase to infinity and the path will cross the horizon and approach to the center. Otherwise...

2. Initially u decreases (with the "-" sign in ODE (5)), and initially the motion is outbound. Being outbound...
3. The photon either reaches its minimum $u_{\min}$ inside area A somewhere between the horizon and circular orbit when $b < b_{crit} = 3GM\sqrt{3}$. After reaching $u_{\min}$, (or $r_{\max}$), the photon moves toward the center being caught.
4. Or it crosses the circular orbit into area A and escapes into infinity when $b > b_{crit}$.
5. Or, if $b = b_{crit}$, the photon asymptotically approaches the circular orbit in a tight infinite spiral.

Inbound only path $r(\varphi)$ in area C

When in area C, the physical theory admits only inbound motion no matter the value b, and only the choice of sign "+" in ODE (5) so that $u = \frac{1}{r}$ increases to infinity.

Tightness of the infinite spiral with $b = b_{crit}$

Here we are to discuss and introduce a measure of tightness of the special infinite spiral capturing a photon when $b = b_{crit}$. Obtain the ODE for $\frac{dr}{d\varphi}$ by dividing the ODEs (2):

$$\begin{aligned}\frac{dr}{d\varphi} &= \pm\frac{r^2}{b}\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2GM}{r}\right)} \\ &= \pm\frac{r^2}{b}\sqrt{1-\frac{b^2}{r^3}(r-2GM)} \\ &= \pm\sqrt{\frac{r^4}{b^2}-r(r-2GM)}.\end{aligned}$$

It's convenient to introduce and study a radial distance $\rho = 3GM - r$, $(r = 3GM - \rho;\quad r' = -\rho')$ measuring the distance of spiral laps in proximity of the circular radius $3GM$.

We need the formula above under the condition that $b = b_{crit} = 3GM\sqrt{3}$ or $b^2 = 27\,(GM)^2$.

$$r^4 = 81\,(GM)^4 - 108\,(GM)^3\,\rho + 54\,(GM)^2\,\rho^2 - 12\,(GM)\,\rho^3 + \rho^4$$

$$\frac{r^4}{b^2} = 3\,(GM)^2 - 4\,(GM)\,\rho + 2\rho^2 - \frac{12}{27\,(GM)}\rho^3 + \frac{\rho^4}{27\,(GM)^2}$$

$$
\begin{aligned}
r(r-2GM) &= (3GM-\rho)(3GM-\rho-2GM) \\
&= (3GM-\rho)(GM-\rho) \\
&= 3(GM)^2-\rho GM-3\rho GM+\rho^2 \\
&= 3(GM)^2-4\rho GM+\rho^2
\end{aligned}
$$

$$
\frac{r^4}{b^2}-r(r-2GM)=\rho^2-\frac{12}{27(GM)}\rho^3+\frac{\rho^4}{27(GM)^2}
$$

and finally we get

$$
\frac{d\rho}{d\varphi}=-\rho\sqrt{1-\frac{12}{27(GM)}\rho+\frac{\rho^2}{27(GM)^2}}. \qquad ((10))
$$

When a photon approached close to the circular orbit, ρ is negligibly small, so that we can simplify the ODE above:

$$
\frac{d\rho}{d\varphi}\approx-\rho
$$

whose solution is

$$
\boxed{\rho\approx e^{-\varphi}.}
$$

When φ grows from 0 to infinity, $r=3GM-\rho$ represents a tight infinite exponential - the solution of ODE (10).

In order to study the tightness of the spiral, consider the sequence $\{\rho_n = \rho((2n+1)\pi)\}, \quad n=1,\ 2,\ldots$ These are points on consecutive laps of the spiral whose distance to the circular radius $\rho_n\to 0$. The tightness of the spiral may be measured by a coefficient $k=\frac{\rho_{n+1}}{\rho_n}=e^{-2\pi}\approx 0.0018$ and that is what we observed in Table 1 during integration of r (until appearance of numerical artefacts cause by a drop of accuracy).