Exploratorium: a library of real time simulations for applied ODEs

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Abstract

This library of simulations utilizes an advanced ODE solver called the Taylor Center capable to integrate initial value problems with high accuracy and to display solutions as real time animation in 2D and 3D stereo (viewable via red/blue glasses).

These unique graphical features and the user friendly interface provide the environment of a virtual laboratory allowing to observe the important dynamic behavior of the problems gathered in the library, and to experiment modifying the initial values - as though during the real life laboratory work.

This first issue of the Exploratorium deals with the various type of motion of a rolling disk (as a special case of the motion of a rigid body).

Rolling disk

Folder: *RollingDisk*.

The ODEs in the mathematical form

The ODEs (156), (159) in [1] obtained by Dmitry Garanin for a rolling disk are:

$$
\begin{aligned}
\ddot{\theta} &= \frac{\left(-L_3 + I_2\dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta - MgR\cos\theta + F_x R\sin\theta\sin\phi}{I_1'} \\
\frac{d}{dt}\left(\dot{\phi}\sin\theta\right) &= \frac{\left(L_3 - I_1'\dot{\phi}\cos\theta\right)}{I_2}\dot{\theta} \\
\dot{L}_3 &= MR^2\dot{\theta}\dot{\phi}\sin\theta - F_x R\cos\phi \\
\dot{x}_c &= R\left(\dot{\theta}\sin\theta\sin\phi - \frac{L_3\cos\phi}{I_3'}\right) \\
\dot{y}_c &= -R\left(\dot{\theta}\sin\theta\cos\phi + \frac{L_3\sin\phi}{I_3'}\right)
\end{aligned}
$$

where

$$
I_1' = \frac{I}{2} + MR^2, \quad I_2 = \frac{I}{2}, \quad I_3' = I + MR^2,
$$
$$
L_3 = I_3'(\dot{\psi} + \dot{\phi}\cos\theta)
$$

We re-wrote them using the expression for L_3 into a system containing all Euler angles $\theta,\ \phi,\ \psi$:

$$
\begin{aligned}
\ddot{\theta} &= \frac{\left(-L_3 + I_2\dot{\phi}\cos\theta\right)\dot{\phi}\sin\theta - MgR\cos\theta + F_x R\sin\theta\sin\phi}{I_1'} \\
\ddot{\phi} &= \frac{L_3 - (I_1' + I_2)\,\dot{\phi}\cos\theta}{I_2\sin\theta}\dot{\theta} \\
\dot{\psi} &= \frac{L_3}{I_3'} - \dot{\phi}\cos\theta \\
\dot{L}_3 &= MR^2\dot{\theta}\dot{\phi}\sin\theta - F_x R\cos\phi \\
\dot{x}_c &= R\left(\dot{\theta}\sin\theta\sin\phi - \frac{L_3\cos\phi}{I_3'}\right) \\
\dot{y}_c &= -R\left(\dot{\theta}\sin\theta\cos\phi + \frac{L_3\sin\phi}{I_3'}\right)
\end{aligned} \tag{1}
$$

Typically the solid body motion equations are written in the rotating system of coordinates [2] which is fixed with the rotating body. Dmitry Garanin transformed them into the laboratory system, where the disk position is expressed via the Euler angles θ, ϕ, ψ (the picture on page 24 in [1]). The meaning of these angles is easy to explain in a case of a uniform motion of a declined disk along a circular line:

θ is the angle of decline of the disk to the plane ($\dot{\theta}$ being a velocity of change of this decline during non-uniform motion);

$\dot{\psi}$ is the angular velocity of the intrinsic rotation of the disk around its axis perpendicular to the disk at its center;

$\dot{\phi}$ is the angular velocity of the node line - the line of intersection of the disk plane and horizontal plane. In particular, if a disk is positioned perpendicularly to the plane and spins around its diameter perpendicular to the plane, $\dot{\phi}$ is the angular velocity of this rotation.

As explained in [1],

$$L_3 = I_3'(\dot{\psi} + \dot{\phi}\cos\theta) \tag{2}$$

is a component of angular momentum.

Remark 1 *In the literature on this topic, the meaning of* $\dot{\phi}$ *and* $\dot{\psi}$ *may be vice versa of that used here and by Garanin in [1].*

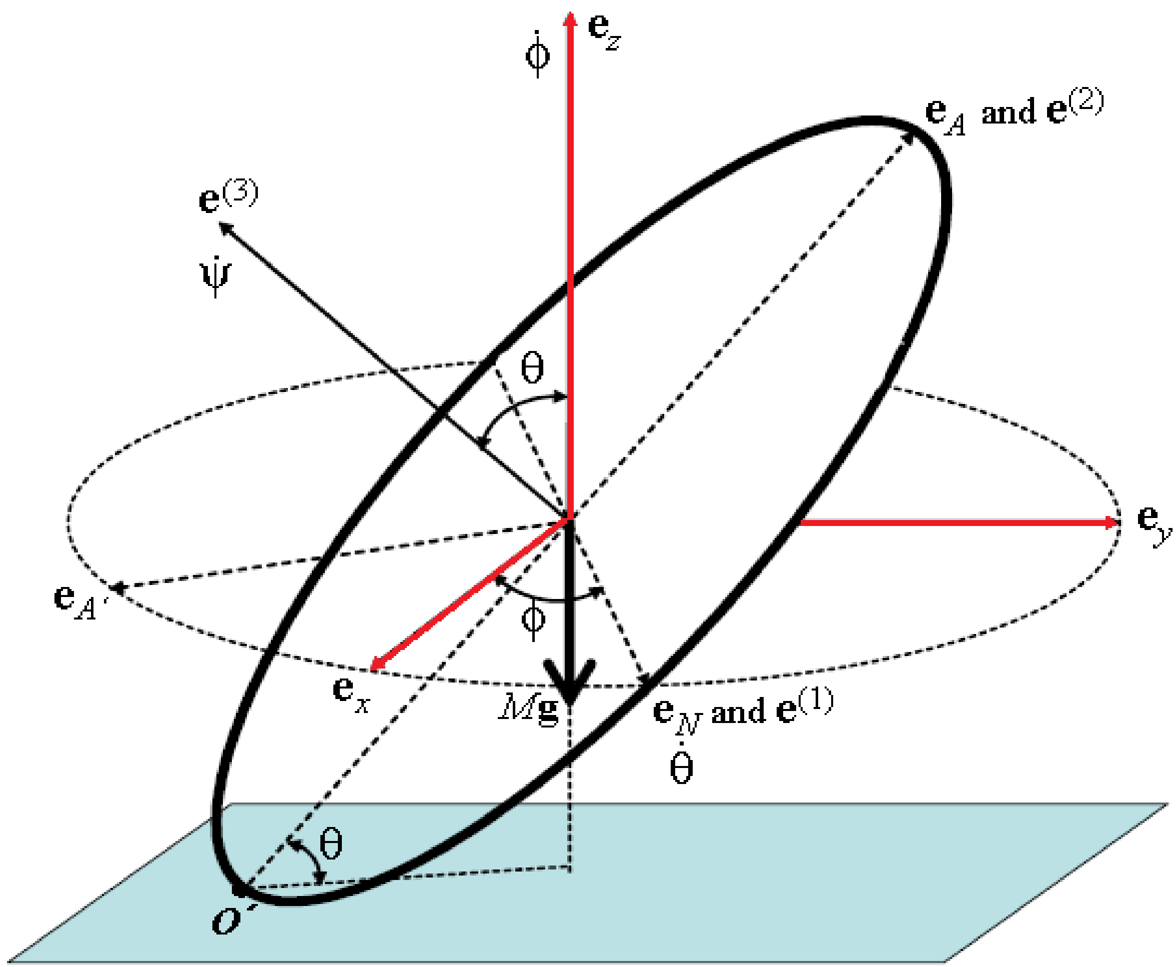
Remark 2 *This system of ODEs requires the initial values for* θ, $\dot{\theta}$, ϕ, $\dot{\phi}$, ψ, L_3, x_c, y_c *at* $t = 0$. *However, considering the known equation for* L_3 *(2),* L_3 *and* $\dot{\psi}$ *are related. If we specify one, we automatically get the other. Therefore in some cases we specify* $L_3|_{t=0}$*, in others it's more convenient to specify* $\dot{\psi}|_{t=0}$ *instead.*

We are going to consider the following groups of cases.

1. Examples visualizing the meaning for each of the angular velocities $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$ (folder *EachAngleActions*);

2. The special cases of uniform motion when θ, $\dot{\phi}$, $\dot{\psi}$ remain constant (folder *UniformMotion*). The subcase when the center of the disk remains still during the motion - the so called Euler disk [4].

3. Examples of interesting dynamic from Dmitry Garanin [1] (folder *GaraninExamples*);



$\dot{\phi}$
$\mathbf{e}_z$
$\mathbf{e}_A$ and $\mathbf{e}^{(2)}$
$\mathbf{e}^{(3)}$
$\dot{\psi}$
θ
$\mathbf{e}_y$
$\mathbf{e}_{A'}$
ϕ
$\mathbf{e}_x$
$M\mathbf{g}$
$\mathbf{e}_N$ and $\mathbf{e}^{(1)}$
$\dot{\theta}$
θ
O

Notation in the software

$\dot{\theta}$	te1
$\ddot{\theta}$	te1'
$\dot{\phi}$	fi1
$\ddot{\phi}$	fi1'
ψ	psi
$\dot{\psi}$	psi1
L_3	L3
x_c	xc
y_c	yc
z_c	zc
I'_1	i11
I_2	i2
I'_3	i31

ζ_x	ksix
ζ_y	ksiy
ζ_z	ksiz
F_x	Fx
F_y	Fy
π	Pi
$\pi/5$	Pi5
$\theta\|_{t=0}$	te0
$\dot{\theta}\|_{t=0}$	te10
$\phi\|_{t=0}$	fi0
$\dot{\phi}\|_{t=0}$	fi10
I'_1	i11
I'_3	i31
$L_3\|_{t=0}$	L30

Method of visualization

We achieve visualization of the moving disk via setting 11 trajectories

$$\begin{array}{l} \{x_0,\ y_0,\ z_0\} \\ \dots \\ \{x_9,\ y_9,\ z_9\} \\ \{\zeta_x,\ \zeta_y,\ \zeta_z\} \end{array}$$

(see the page *Graph setting* in the Main window of the program). Here $\{x_i(t),\ y_i(t),\ z_i(t)\}$ represent 10 evenly placed fixed points at the edge of the disk. For them only their motion is displayed not plotting the trace of the motion (no plotted trajectories), while $\{\zeta_x,\ \zeta_y,\ \zeta_z\}$ represents the trajectory of the Contact Point, and this trajectory is plotted. (This is achieved due to the parameter *Lines and solid body/Plot beginning with the curve* # 11).

Equations in the software

Script file: *L3Declined1.scr*

The constants.

Pi = 3.141592653589793238462643
p5 = Pi/5
M = 1
g=100
R = 1
MR2 = M*R^2

MgR = M*g*R
i = 1
Fx = 0
te0 = 0.5
te10 = 0
i2 = i/2
fi0 = Pi/2 {fi(0)}
i11 = i/2 + MR2 {i1'}
i31 = i + MR2 {i3'}
L30 = 10
fi10 = (L30 - sqrt(L30^2 + 4*i2*MgR * (cos(te0))^2/sin(te0))) / (2*i2*cos(te0))

The Initial values

t = 0
te = te0
fi = fi0
Psi = 0
te1 = te10 {te'(0)}
fi1 = fi10 {fi'(0)}
L3 = L30
xc = 0
yc = 0

Auxiliary variables

coste = cos(te)
sinte = sin(te)
cosfi = cos(fi)
sinfi = sin(fi)
cospsi = cos(psi)
sinpsi = sin(psi)
cosficoste = cosfi*coste
sinficoste = sinfi*coste
te1sinte = te1*sinte
fi1coste = fi1*coste
zc = R*sinte {Center of mass z}
ksix = xc + R*coste*sinfi {Contact point on plane x }
ksiy = yc - R*coste*cosfi {Contact point on plane y}
ksiz = 0 {Contact point on plane z}
cospsi1 = cos(psi + p5)
sinpsi1 = sin(psi + p5)
cospsi2 = cos(psi + p5*2)
sinpsi2 = sin(psi + p5*2)
. . . . .
cospsi9 = cos(psi + p5*9)

sinpsi9 = sin(psi + p5*9)
x0 = xc + R*(cosfi*cospsi - sinficoste*sinpsi) {a fixed point at the edge of the disk, x}
y0 = yc + R*(sinfi*cospsi + cosficoste*sinpsi) {a fixed edge point y}
z0 = zc + R*sinpsi*sinte {a fixed point at the edge of the disk, z}
. . . . .
x9 = xc + R*(cosfi*cospsi9 - sinficoste*sinpsi9)
y9 = yc + R*(sinfi*cospsi9 + cosficoste*sinpsi9)
z9 = zc + R* sinpsi9*sinte

ODEs

t'=1
te' = te1
fi' = fi1
Psi' = L3/i31 - fi1coste
te1' = ((i2*fi1coste - L3)*fi1*sinte - MgR*coste + Fx*R*sinte*sinfi)/i11
fi1' = (L3 - (i11 + i2)*fi1coste)*te1/(i2*sinte)
L3' = MR2*te1sinte*fi1 - Fx*R*cosfi
xc' = R*(te1sinte*sinfi - L3*cosfi/i31) {Center of mass x}
yc' = -R*(te1sinte*cosfi + L3*sinfi/i31) {Center of mass y}

Visualizing the meaning for each of the angular velocities $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$

(folder *EachAngleActions*)

The meaning of $\dot{\psi}$.

Let's begin with the intrinsic rotation of the disk with velocity $\dot{\psi}$. To isolate this motion, we set the declination of the disk $\theta|_{t=0} = \pi/2$ (perpendicularly to the plane) and $\dot{\phi}|_{t=0} = \dot{\theta}|_{t=0} = 0$.

Load script file *PsiOnlySpinVert.scr* and click the *Play* button.

Watch the disk rolling with the angular velocity $\dot{\psi} = const = 10$ (psi10=10) in vertical position with uniform speed along the straight line rolling.

If the initial declination of the disk $\theta|_{t=0}$ is not perpendicular, even with initial velocities $\dot{\phi}|_{t=0} = \dot{\theta}|_{t=0} = 0$ they will not remain zero (which we want in order to watch only $\dot{\psi}$ effect).

We cannot set declination $\theta|_{t=0} = 0$ because $\sin\theta$ is in the denominator of the equation for fi1'. We can watch however the motion when $\theta|_{t=0}$ is near zero.

Load script file *PsiOnlySpinNearHoriz.scr* and click the *Play* button.

Watch the disk rolling near horizontally with intrinsic angular velocity $\dot{\psi}$ near 30 along a circular curve (though not a circle). The declination of the disk slightly varies. In order to see that, click *Graph setting* tab in the main window. While there, click *2D* button (which clears the set of trajectories). Now specify the curve te(t). In order to do it, first click intersection of t with X axis, and then click the intersection of te with Y axis. That would set the curve {t, te}. Now click *Graph*. It will show you a sine-like wave of variation of the declination between 0.08 and 0.1 of radian.

The meaning of $\dot{\phi}$.

This is velocity of precession, i.e. angular velocity of the node line. To isolate this motion, we set the declination of the disk $\theta|_{t=0} = \pi/2$ (perpendicularly to the plane) and $\dot{\psi}|_{t=0} = \dot{\theta}|_{t=0} = 0$.

Load script file *FiOnlySpin.scr* and click the *Play* button.

Watch the disk spinning around the vertical diameter with the angular velocity $\dot{\phi} = const = 10$ (fi10=10).

The meaning of $\dot{\theta}$.

This is velocity of the change of declination θ of the disk. To isolate this motion, we must set $\dot{\psi}|_{t=0} = \dot{\phi}|_{t=0} = 0$. In doing so, if we leave declination of the disk $\theta|_{t=0} = \pi/2$ perpendicularly to the plane, no motion may happen as this is a position of equilibrium (though unstable). In order to trigger the motion, we must either provide a small push $\dot{\theta}|_{t=0} = 0.01$, or set the initial declination of the disk slightly less than $\pi/2$. Let's try both.

Load script file *TetaOnlyFall.scr* (with a small push $\dot{\theta}|_{t=0} = 0.01$) and click the *Play* button.

The disk starts falling slowly, accelerates, and ... What?! It falls through the plane as though the plane didn't exist, continuing its motion as though a rigid pendulum.

You can see the same paradox loading script file *TetaOnlyFallPendulum.scr* (with a $\theta|_{t=0} = \pi/2 - 0.01$, all angular velocities being zero). Again you will see the behavior of rigid pendulum, but now swinging back and force disregarding the plane.

The explanation of this paradox is that with $\dot{\psi} \equiv \dot{\phi} \equiv 0$ (and $F_x = 0$) the first ODE of the system (1) really turns into the equation of a pendulum

$$\ddot{\theta} = -\frac{MgR\cos\theta}{I_1'} \tag{3}$$

written for an angle complementary to $\pi/2$ so that we have cosine instead of sine (as in the standard pendulum ODE). The ODE (3) describes also the motion of a disturbed vertical stick or vertical ladder falling down presuming that such a stick-like object is fixed with a hinge as a pendulum and moves disregarding "obstacles" such as horizontal plane.

Now load script file *TetaOnlyFallWithSmallFi.scr* where we added only a negligible initial angular velocity $\dot{\phi}|_{t=0} = 0.001$ around the vertical diameter of the disk: can such a negligible addition change the "free pendulum" behavior of the disk which falls not "noticing" the plane? Click the *Play* button.

Amazingly, now the disk does not fall and swing as a pendulum any more, but fully "acknowledges" the existence of the plane, rolling along a cycloid-like curve: almost flat falling and then standing up again, on and on!

Load *TetaOnlyFallWithSmallFiLong.scr* and watch a longer period of such motion noticing that the cycloid-like curve outlines some large circle.

Well, then how does a small intrinsic spin $\dot{\psi}$ affects the free fall of the disk?

Load script file *TetaOnlyFallWithSmallPsi.scr* where we added only a negligible initial angular velocity $\dot{\psi}|_{t=0} = 0.001$ around the axis of the disk: can this negligible addition change the "free pendulum" behavior of the disk ignoring presence of the plane? Click the *Play* button.

Now the disk also does not fall through as a pendulum any more. Again it "acknowledges" the existence of the plane, but rolls along a sine-like curve: almost flat falling and then standing up again, on and on!

Load *TetaOnlyFallWithSmallPsiLong.scr* and watch a longer period of such motion noticing that now this sine-like curve outlines a straight line (rather than circle).

Conclusion 3 *If the disk in the initial (near) vertical position has zero angular velocities $\dot{\psi}$ and $\dot{\phi}$ so that its interaction with the plane takes place only via one point at the initial position, there are no forces which change its free fall as a rigid pendulum which ignores existence of the plane. However, one of the velocities $\dot{\psi}$ or $\dot{\phi}$ being nonzero does cause the disk to move away from the initial point of touch along some curve. It's the forces of reaction of the plane at points of this curve which turn the disk changing its motion so that the disk stays above the plane touching it along the curve of touch.*

One more observation of a technical nature.

Remark 4 *In the case TetaOnlyFall.scr of free fall, the declination θ, beginning with $\pi/2$, goes down bypassing the zero value. However, the 2nd of the ODEs (1) contains $\sin\theta$ in the denominator. In most cases an attempt to integrate bypassing a point of singularity of the ODE fails no matter whether the solution is regular or not at this point - and with the initial values of this case the solution is regular when $\theta = 0$. It is because of the regularity of the solution and small values of its Taylor coefficients the program applies such a large integration step, that by mere chance it bypasses the point where $\theta = 0$ so that the integration succeeds.*

The special cases of uniform motion when $\theta,\ \dot\phi,\ \dot\psi$ remain constant

(folder *UniformMotion*)

Intuitively we can imagine such a physical setting that the disk rolls along the circumference with constant decline $\theta < 90^\circ$ and constant angular velocities $\dot\phi,\ \dot\psi$ in such a way that the centrifugal force is compensated with the horizontal component of weight force. We are to demonstrate, that such a solution of the system (1) does exist. We assume, indeed, that $F_x = 0$ (no external horizontal forces).

Observe that if $\theta(t) = const,\ \dot\theta = 0$ and $\dot\psi = 0$ satisfy the system, then also $L_3 = I_3'(\dot\psi + \dot\phi\cos\theta) = const$. With that in mind, we see that the equations for $\ddot\phi,\ \dot\psi$, and $\dot L_3$ are satisfied having zero right hand sides. As to the ODE for $\ddot\theta$, let's set a condition that the right hand side be zero:

$$\left(-L_3 + I_2\dot\phi\cos\theta\right)\dot\phi\sin\theta - MgR\cos\theta = I_2\dot\phi^2\cos\theta\sin\theta - L_3\dot\phi\sin\theta - MgR\cos\theta = 0.$$

Solving this square equation in $\dot\phi$, we get

$$\dot\phi = \frac{L_3\sin\theta \pm \sqrt{L_3^2\sin^2\theta + 4I_2MgR\cos^2\theta\sin\theta}}{2I_2\cos\theta\sin\theta}$$

or

$$\dot\phi = \frac{L_3 \pm \sqrt{L_3^2 + 4I_2MgR\cos^2\theta/\sin\theta}}{2I_2\cos\theta}.$$

Given the fact that all the values in the right side are constants, so is also $\dot\phi$. Therefore the set of constant functions $\theta,\ \psi,\ L_3$ and then the constant $\dot\phi$ computed based on the values of the former, do satisfy the system of (1) of the source ODEs. Further on, in the examples illustrating this uniform motion, we will arbitrarily choose the constant values for θ and angular momentum L_3 obtaining the rest of the initial values by the known formulas.

An interesting sub-case of this uniform motion is the case when the center of the mass remains still during the disk motion. Examining the ODEs (1) for $\dot x_c,\ \dot y_c$, we see that they may be zero only if the constant $L_3 = 0$. That means that in order to extract the Euler case from the general case of uniform motion, we must set $L_3 = 0$ and then getting

$$\dot\phi = \frac{\pm\sqrt{4I_2MgR\cos^2\theta/\sin\theta}}{2I_2\cos\theta} = \pm\sqrt{\frac{MgR}{I_2\sin\theta}}.$$

That's the initial constant value for $\dot\phi$ depending on the decline θ for the Euler case.

To watch the discussed above cases of the uniform motion, from the folder *UniformMotion...*

- Load and play script files *L3Declined1.scr*, *L3NearVert1.scr*, *L3NearVert2.scr* displaying uniform motion with the center of the disk moving along a circle;

- Load and play script files *Euler45.scr*, *EulerNear90.scr*, *EulerNearFlat.scr* displaying uniform motion with the center of the disk resting.

See the implementation of the formulas above in the section of *Constants* under the tab *Equation setting* in the front window.

Examples from Dmitry Garanin in [1]

(folder *GaraninExamples*)

The following exaplanation of the examples comes from the section 4.1.3 in [1].

Example 1. The parameters of the wheel are set to $M = R = I = 1$. The results show that a rolling

wheel never falls on a side in spite of the gravity torque. Rolling with the rotation around the symmetry axis is pretty stable. Applied force F_x tends to accelerate the wheel's motion in the direction of the force. If the wheel's initial rotation is very slow, it nearly falls flat but, as θ approaches 0 or π, the rotation dramatically increases so that both $\dot{\phi}$ and $\dot{\psi}$ become large and the sign of $\dot{\theta}$ gets reversed. During the short nearly-fall time interval both the Center of Mass (CM) and the contact point assume large velocities and the contact point makes a bow around the CM. Then the wheel stands up again until the next fall on one of the sides.

One of the numerical solutions in the nearly falling regime is presented below.

Load and play the script file *NearFlatFall1.scr*.

The forces are $Mg = 100$ and $F_x = 0$. The initial conditions were $x_c(0) = y_c(0) = 0$, $\theta(0) = \pi/2$, $\phi(0) = \pi/2$ and $\dot{\theta}(0) = 0.01$, $\dot{\phi}(0) = 0$, $\dot{\psi}(0) = -0.001$. That is, in the initial state the wheel is upright, its plane is parallel to the y-axis and it begins rolling very slowly in the positive y direction because $\dot{\psi}(0) < 0$. However, a small push in the positive x direction, $\dot{\theta}(0) = 0.01$, together with the gravity torque cause the wheel to nearly fall flat. The CM begins to move to the right and almost reaches the surface. But because of the small initial rotation the derivatives $\dot{\phi}$ and $\dot{\psi}$ strongly increase and the wheel making a fast rotation with displacement stands up again.

In order to see the changes of the θ, ϕ, ψ load and play the script file *NearFlatFall1TeFiPsi*.scr . It demonstrates $\theta(t)$ nearly falling and recovering. ϕ shows that the wheel is rapidly precessing in alternating directions when the wheel nearly hits the ground, while ψ demonstrates at those moments how the wheel is rapidly rotating around its symmetry axis too.

Let us discuss now the motion of of the Center of the Mass (CM) and of the Contact Point (CP).

In order to watch the vertical motion of the CM, in the main window under the tab *Graph setting* click *Clear* and specify the curve {t, zc} by clicking first intersection of t and X axis, and then zc and Y axis. Click *Graph*. Then the Graph window will display $Z(t)$ and you may *Play* it to see the dynamic.

Then, in order to watch the horizonal motion of the CM and CP load and play the script file *NearFlatFall1CMCPi.scr .* One can see that as the wheel nearly touches the ground CP makes a half-circle around CM.

Example 2. This is the same as Example 1 except that now precession[1] $\dot{\phi}(0) = 0.25$. This initial precession causes the CM of the wheel to make circles.Load and play the script file *NearFlatFall2.scr* . $\dot{\phi}(0) > 0$ also makes the CP motion of a cycloid type.

Load and play the CM-CP curves only: *NearFlatFall2CM-CP.scr* .

Example 3. This is the same as Example 1 but adding also a horizontal force $F_x = 0.03$ (while leaving $\dot{\phi}(0) = 0.25$ as in Example 2). Load and play *NearFlatFall3.scr* : the external force makes the CM motion of a

cycloid type. Watch it also in a plane version: *NearFlatFall3CM-CP.scr .*

Example 4. Reversing the sign of the force $F_x = -0.03$ (with the same $\dot{\phi}(0) = 0.25$) completely changes the character of the motion. Load and play *NearFlatFall4.scr .*

With a longer time interval, one can see the change of regime after some time - script file *NearFlatFall4Longer.scr* (page 35 in [1]).

Example 5. With all the same but the faster precession $\dot{\phi}(0) = 1$, the motion becomes even more complecated - *NearFlatFall5.scr .*

References

1. Dmitry Garanin. "Rotational motion of rigid bodies". 2004, http://lehman.edu/faculty/dgaranin/Mechanics/Mechanis_of_rigid_bodies.pdf

2. Euler's equations (rigid body dynamics), https://en.wikipedia.org/wiki/Euler%27s_equations_(rigid_body_dynamics)

3. https://en.wikipedia.org/wiki/Rigid_body_dynamics .

4. Euler disk https://en.wikipedia.org/wiki/Euler%27s_Disk

[1] In the original article [1] $\dot{\phi}(0) = 0.1$ and I have no clarifications from the author.