The Unifying View on Ordinary Differential Equations and Automatic Differentiation, yet with a Gap to Fill

Alexander Gofen

Abstract. Among various instruments of approximation representing solutions of Ordinary Differential Equations (ODEs), the Taylor expansions play a unique role. This is because an explicit system of ODEs by itself is a tool for computing $n-$order derivatives of the solution and therefore delivering the Taylor expansions of the solution at multiple points of the phase space.

This capability by ODEs to generate Taylor expansions relies however on availability of the rules of $n-$order differentiation and their computational efficiency. As shown in [2], availability and efficiency of the rules of $n-$order differentiation take place only for a particular sub-class of holomorphic functions. Those are the so called *generalized elementary* functions [1, 2], i.e. functions representable as solutions of rational ODEs.

It was shown [3], that continuation of (generalized) elementary functions via integration of its ODEs not necessarily expands them into each and every point where these functions exist and are holomorphic. Some entire functions are suspects for being elementary everywhere except isolated unreachable points - the points of their "removable" or "regular" singularity [3].

All the above mentioned and a few other issues are interdependent and complementary to each other. The merger of them into one theory is called the *Unifying view on ODEs and AD* [2].

However there is a gap in this otherwise coherent view: an open statement, the Conjecture [2,3] about the possibility to convert a rational system of ODEs at a regular point into one $n-$order rational ODE regular at the same point. The Conjecture is important because the question of equivalency of two competing definitions of elementary functions (and a few other open statements) depend on the Conjecture.

The report therefore presents the setting and a few known facts concerning the Conjecture as an invitation for everybody to resolve it and fill the gap in this theory.

1. Preface

Typically, ODEs are considered for classes of real valued $n-$times differentiable functions. However all along this report we deal with ODEs over a class of *holomorphic* functions (for the reasons explained below). This implies derivatives understood as the complex derivatives in the complex plane $\mathbf{C}$, (rather than derivatives along the real axis). However we do not mention explicitly the complexity

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of the functions we deal with, because operations over them in the complex plane **C** formally look as though on the real axis.

Among various instruments of approximation representing solutions of ODEs, the Taylor expansions play a unique role. This is so because an explicit system of ODEs by itself is a tool for computing $n-$order derivatives of the solution, therefore delivering the Taylor expansions of the solution at multiple points of the phase space.

The plurals "expansion*s*" and "multiple point*s*" here are crucial. A holomorphic function represented only by an arbitrary Taylor expansion at only one point in theory may be analytically continued, yet usually not in practice. The Taylor expansion of a function at one point does not help to obtain the expansion at the other points. It is ODEs that generate the Taylor expansion at any regular point of the phase space thus making the analytical continuation possible and practical.

This capability to generate Taylor expansions by ODEs relies however on availability of the rules of $n-$order differentiation and their computational efficiency. As shown in [2], the rules of $n-$order differentiation are available and efficient only for a particular sub-class of all holomorphic functions. Those are functions representable by rational ODEs and called *generalized elementary* functions [1,2]. They widen the class of conventional elementary functions so that the class becomes closed. In terms of a generalizing definition, the solutions of elementary ODEs are elementary too.

It was shown [3], that continuation of (generalized) elementary functions via integration of its ODEs not necessarily expands them into each and every point where these functions exist and are holomorphic. Some entire functions are suspects for being elementary everywhere except an isolated point: the point of their "removable" or "regular" singularity [3]. This gives another insight into the notion "removable singularity" as a new type of special point in holomorphic functions, which is rather "unmovable", being proper to a particular holomorphic function.

The following ideas at first glance may seem unrelated:

- Elementary functions;
- The class of elementary ODEs closed with regard to their solutions;
- The transformations of all elementary ODEs to the special formats, enabling...
- Optimized computability of $n-$order derivatives, enabling...
- The modern Taylor method as a tool of efficient analytic continuation;
- Special points unreachable via integration of ODEs - points of violation of elementariness.

They all however are interdependent and complementary to each other. An approach revealing the merger of these concepts is important enough by itself, so that we call it the *Unifying view on ODEs and AD* [2] framed by the theory of holomorphic functions.

There is a gap in this otherwise coherent view: an open statement, the Conjecture [2,3] about possibility to convert a rational system of ODEs at a regular point into one $n-$order rational ODE regular at the same point. The Conjecture is so important because two other open statements depend on it:

- The elementariness may be defined similarly either for vector-functions via rational systems of 1st order ODEs, or for a stand alone function via one $n-$order ODE. Both definitions would be equivalent only if the Conjecture were proved;

- The recently discovered specialty of points like $t = 0$ in the function $x(t) = \dfrac{\sin t}{t}$ (and others [3]) shows up in that $x(t)$ can not satisfy any rational $n-$order ODE regular at $t = 0$. Is it also true, that $x(t)$ can not satisfy any rational *system* of ODEs regular at $t = 0$? It would be true if the Conjecture were proved.

The report therefore presents the setting and a few known facts concerning the Conjecture as an invitation for everybody to resolve it.

2. The Definitions

Since R. Moore [1], the concept of (generalized) elementary functions was introduced for vector-functions, and their definition relied on a system of rational ODEs [2].

DEFINITION 1. *A vector-function* $\mathbf{f} : \mathbf{C}^m \to \mathbf{C}^n$, $\mathbf{f} = \{f_k(x,\ y,...)\}$, $k = 1, \dots,\ n$, *is called elementary in a certain variable (say x) near an initial point* $(a,\ b,...)$ *if* $\{f_k(x,\ y,...)\}$ *is a solution of some initial value problem for a system of $N \geq n$ autonomous* ordinary *differential equations*

$$(2.1)\qquad \left\{ \frac{\partial u_k}{\partial x} = R_k(u_1, ...,\ u_n, ...,\ u_N); \qquad u_k|_{x=a} = f_k(a,\ b, ...), \quad k = 1, \dots,\ N \right.$$

with rational right hand sides R_k regular *at the initial point. The components u_k are viewed as functions of x, all the remaining variables being considered as parameters.*

EXAMPLE 1. *A function $f(x,\ y) = \cos(x)e^y$ as a function of x is elementary at a point $x = a,\ y = b$ because $f(x,\ y)$ satisfies a system*

$$\begin{aligned} f'_x &= -g; \quad f|_{x=a} = \cos(a)e^b \\ g'_x &= f; \quad\ \ g|_{x=a} = \sin(a)e^b \end{aligned}$$

(omitting $x' = 1$). As a function of y, $f(x,\ y)$ is elementary because it satisfies a system

$$\begin{aligned} f'_y &= f; \quad f|_{y=b} = \cos(a)e^b \\ g'_y &= 0; \quad g|_{y=b} = const \quad \text{('dummy' component)} \end{aligned}$$

(omitting $y' = 1$).

Therefore elementariness of an $n-$dimensional vector-function (or of a stand alone function) sometimes may be established only via a system of ODEs with number of equations $N > n$, like elementariness in x of $f(x,\ y) = \cos(x)e^y$ in the example above. We then say that a (vector-)function is elementary together with the associated components: say $\cos t$ together with $\sin t$.

Is it possible to define a "stand alone" elementariness of a component of a vector-function not referring to the associated components so that the "stand alone" elementariness be equivalent to Definition 1 in a certain sense?

Consider a definition of hypothetical elementariness for a stand alone function $f(x_1, ...,\ x_m)$.

Definition 2. *A function* $f : \mathbf{C}^m \to \mathbf{C}^1,\ f(x,\ y, ...)$ *is called "stand alone" elementary in a variable* x *near an initial point* $(a,\ b, ...)$ *if* $f(x,\ y, ...)$ *is a solution of some initial value problem for an* $N-$*order* ordinary *differential equation in* x

(2.2)

$$\left\{u^{(N)} = R(x,\ u,\ u', ...,\ u^{(N-1)}); \quad u^{(k)}\Big|_{x=a} = \frac{\partial^k f(a,\ b, ...)}{\partial x^k}, \quad k = 0, \ldots,\ N-1\right.$$

with rational right hand side R *regular at the initial point.*

Example 2. *A function* $f(x,\ y) = \cos(x)e^y$ *as a function of* x *is elementary at a point* $x = a,\ y = b$ *because* $f(x,\ y)$ *satisfies an ODE*

$$f''_x = -f; \quad f|_{x=a} = \cos(a)e^b, \quad f'|_{x=a} = -\sin(a)e^b.$$

As a function of y, $f(x,\ y)$ *is elementary because it satisfies an ODE*

$$f'_y = f; \quad f|_{y=b} = \cos(a)e^b.$$

An issue of equivalency between Definitions 1 and 2 relies on the general concept of *equivalent transformations* between a system of m 1st order ODEs (2.4) and one $n-$order ODE (2.3). The concept of *equivalent transformations* applies to arbitrary holomorphic (rather than rational) ODEs.

Definition 3. *("One ODE → System" transformation). We say that one* $n-$*order ODE (2.3) equivalently transforms into a system of* m *1st order ODEs (2.4) in respect to* u_1 *at a regular point* t_0 *if the solution* u_1 *of an ODE*

$$u_1^{(n)} = f(t,\ u_1, ...,\ u_1^{(n-1)}), \quad u_1^{(i)}\Big|_{t=t_0} = a_1^i, \quad i = 1, ...,\ n-1 \tag{2.3}$$

belongs to a solution vector of a regular IVP for a 1st order system

$$\{u'_k = g_k(t,\ u_1, ...,\ u_m)\ , \quad u_k|_{t=t_0} = b_k, \quad b_1 = a_1^0, \quad k = 1, ...,\ m. \tag{2.4}$$

Definition 4. *("System → One ODE" transformation). We say that a system of* m *1st order ODEs (2.4) equivalently transforms into one* $n-$*order ODE (2.3) in respect to a particular variable (say* u_1*) if the component* u_1 *of the solution of system (2.4) satisfies a regular IVP for one* $n-$*order ODE (2.3).*

It is obvious that Definitions 1 and 2 of elementariness are equivalent if the equivalent transformations "*System → One ODE*" and "*One ODE → System*" are possible for rational right hand sides $f,\ g_k$ (see Table 1, cell "?"). The case of holomorphic (rather than rational) right hand sides is brought into the Table 1 just for comparison: to illustrate that this case is trivial (proven below) so that only the case in the cell "?" stands on the way. This unsolved problem was posed in 2008 as *the Conjecture* [3] and it is analyzed more below.

		$f,\ g_k$ are...	
		Rational	Holomorph
One n-order ODE →	Syst. of m 1^{st} order ODEs	Yes	Yes
Syst. of m 1^{st} order ODEs →	Regular n-order ODE	**?**	Yes
	Singular n-order ODE	Yes	

Table 1. Equivalent transformations "*System → One ODE*" and "*One ODE → System*"

THEOREM 1. *The equivalent transformation "One ODE → System" (Row 1 of Table 1) is possible (and trivial) for both holomorphic and rational right hand sides* $f,\ g_k$.

PROOF. The equivalent transformation of *one* $n-$*order ODE* (2.3) into *a system of 1st order ODEs* is trivially achievable via introduction of new variables $u_{k+1} = u_k' = u_1^{(k)},\ \ k = 1,\ ...,\ n-1$:

$$\begin{cases} u_k' = u_{k+1}\ , & k = 1, ...,\ n-1 \\ u_n' = f(t,\ u_1,\ u_2, ...,\ u_{n-1}). \end{cases}$$

If the function f is rational and regular at $t = t_0$ so is the system. □

The opposite case is not trivial at all.

THEOREM 2. *For* arbitrary holomorphic *right hand sides* $f,\ g_k$ *the equivalent transformation "System → One ODE" is possible and trivial (remaining however an open question for* rational *right hand sides).*

PROOF. Consider the solution $u_1(t)$ of system (2.4), denoting $f(t) = u_1'(t)$. Then the equation $u_1'(t) = f(t)$ is a required equivalent ODE (2.3), and the ODE of 1^{st} order at that. □

REMARK 1. *Without the requirement that the target be a* regular $n-$*order ODE at* $t = t_0$ *, the transformation "System → One ODE" is possible by means of resultants and elimination of variables in certain polynomial equations (5.2), as shown in [2] and discussed below.*

Further on, speaking about elementary functions, we mean elementariness only according to *Definition 1.* As noted in [2], the fundamental theorems about the elementary functions (in particular on their closedness), and the remarkable transformations for system of elementary ODEs were established relying on *Definition 1.* (With Definition 2 the methods of proof applied in [2] would not work).

3. The remarkable transformations

Here we are summarizing the facts (established in [2]) that all explicit first order systems of ODEs with elementary right hand sides

$$\{u_k' = f_k(u_1, ...,\ u_m)\ , \qquad u_k|_{t=t_0} = a_k, \qquad k = 1, ...,\ m \tag{3.1}$$

are convertible to larger systems in special formats outlined in the Table 2 below.

<table>
<tr><td colspan="3">(1) An explicit 1st order system of ODEs whose right-hand sides are elementary vector-functions converts to…</td></tr>
<tr><td colspan="3">(2) A system of ODEs whose right-hand sides are rational functions. It further converts to…</td></tr>
<tr><td rowspan="3">(3) A canonical system: an explicit system of algebraic and differential equations for computing n-order derivatives requiring $O(n^2)$ operations.</td><td colspan="2">(4) A system, whose right-hand sides are polynomials. It further converts to…</td></tr>
<tr><td colspan="2">(5) Polynomial ODEs of degree $\le$ 2. It further converts to polynomial ODEs of degree 2 with ...</td></tr>
<tr><td>(6) ...with coefficients 0, 1 only (Kerner [4])</td><td>(7) ... with squares only (Charnyi [5])</td></tr>
</table>

Table 2. Remarkable transformations of elementary systems of ODEs

We speak about conversion of the source system into a larger target system of ODEs in the sense of the equivalent transformations. That means that the solution vector of the target system contains the solution vector of the source one. Some components of the target system are introduced as known relations over the source components. Therefore the initial values of the target systems are not all free. The newly introduced components are functions of the source initial values, while the corresponding relations are in fact the integrals of the target system.

4. The Conjecture

Conjecture 1. *Consider an IVP for a source system of m rational ODEs*

$$
\begin{aligned}
x' &= \frac{P_1(t,\ x,\ y,\ z, ...)}{Q_1(t,\ x,\ y,\ z, ...)} \\
y' &= \frac{P_2(t,\ x,\ y,\ z, ...)}{Q_2(t,\ x,\ y,\ z, ...)} \\
& \ldots\ldots\ldots\ldots
\end{aligned}
\tag{4.1}
$$

with the denominators $Q_i|_{t=0} \neq 0$ so that the IVP is regular and has a unique solution $(x,\ y,\ z, ...)$ near $t = 0$, in particular the derivatives $x^{(n)}|_{t=0} = a_n$, $n = 0, 1, 2, ...$. Then there exists an explicit rational ODE of some order n

$$
x^{(n)} = \frac{F(t,\ x, ...,\ x^{(n-1)})}{G(t,\ x, ...,\ x^{(n-1)})}; \qquad x^{(k)}|_{t=0} = a_k, \qquad k = 1, 2, ...,\ n-1 \tag{4.2}
$$

with the denominator $G|_{t=0} \neq 0$ so that this ODE is regular at $t = 0$ and has $x(t)$ as its unique solution.

The Conjecture claims convertibility of an explicit first order system of m rational ODEs regular at a given point into one explicit rational ODE of order n regular at this point.

According to Table 2, any *rational* system of ODEs regular at a point may be transformed into a *polynomial* system of degree 2, and then into - a *polynomial* system in squares only

$$\left\{ u_k' = \sum_{i=1}^{m} a_{ki} u_i^2 \ , \quad u_k|_{t=0} = b_k, \ \ k = 1, ..., \ m. \right. \tag{4.3}$$

or in a vector form

$$\mathbf{u}' = A\mathbf{v}, \qquad A = \|a_{ki}\|_{k,i=1}^{m}, \qquad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ ... \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} u_1^2 \\ u_2^2 \\ ... \end{pmatrix}.$$

If an operator of differentiation $\dfrac{d^n}{dt^n}$ is applied, the $n-$times differentiated system takes form

$$\mathbf{u}^{(n+1)} = A\mathbf{v}^{(n)}, \qquad \mathbf{v}^{(n)} = \begin{pmatrix} \sum_{i=0}^{n} C_n^i u_1^{(i)} u_1^{(n-i)} \\ \sum_{i=0}^{n} C_n^i u_2^{(i)} u_2^{(n-i)} \\ ... \end{pmatrix}$$

It is enough to prove the Conjecture assuming that the source system (4.1) is in fact polynomial of any of the special types specified above.

The square only format (4.3) seems especially promising. Assuming the source system being in the format (4.3), the Conjecture was proven [2] for $m = 2$.

Conjecture 2. *For every component (say u_1) of the IVP for polynomial system in squares only (4.3) there exists a regular IVP for $n-$order rational ODE (4.2) having u_1 as a unique solution.*

Proof. (For $m = 2$ only). Consider a system

$$\begin{aligned} x' &= a_1 x^2 + b_1 y^2 \\ y' &= a_2 x^2 + b_2 y^2. \end{aligned}$$

If $b_1 = 0$, the target ODE is $x' = a_1 x^2$. Now assume $b_1 \neq 0$.

$$\begin{aligned} x' &= a_1 x^2 + b_1 y^2, \qquad y^2 = \frac{x' - a_1 x^2}{b_1} \\ y' &= a_2 x^2 + b_2 y^2, \qquad y' = a_2 x^2 + \frac{b_2}{b_1}(x' - a_1 x^2). \end{aligned}$$

Looking at y', observe that all derivatives $y^{(n)}$ depend on x and its derivatives only, and they may be expressed as polynomials G_n: $y^{(n)} = G_n(x, \ x', ... \ x^{(n)})$, $n = 1, 2, ...$ Utilizing this, differentiate the first equation:

$$\begin{aligned} x^{(n+1)} &= a_1 (x^2)^{(n)} + b_1 (y^2)^{(n)} = \\ &= a_1 (x^2)^{(n)} + 2b_1 (y y^{(n)} + C_n^1 y' y^{(n-1)} + ...) \\ &= a_1 (x^2)^{(n)} + 2b_1 (y G_n + C_n^1 G_1 G_{n-1} + ...), \quad n = 1, 2, ... \end{aligned}$$

Now observe that y occurs only in one monomial: that with the factor $G_n(x, \ x', ... \ x^{(n)})$. If at least one $G_n|_{t=t_0} \neq 0$, y may be eliminated and substituted in the following equations, so that we can obtain infinitely many rational ODE's in x regular at $t = t_0$. Otherwise, if all $G_n|_{t=t_0} = y^{(n)}|_{t=t_0} = 0$, $n = 1, 2, ...,$ then $y(t)$ must be a constant meaning the first ODE includes only the dependent variable $x(t)$. That concludes the proof. □

Unfortunately, for $m > 2$ this method of proof does not work, nor does it work for $m = 2$ with addition of mixed or linear monomials in the ODEs.

REMARK 2. *The statement that the target ODE (4.2) is* rational *in the Conjecture is crucial. The following modified Conjecture is false.*

CONJECTURE 3. *(False) Consider a regular IVP for a source system of m rational ODEs (4.1). Then there exists an explicit* polynomial *ODE of some order n*

$$x^{(n)} = P(t,\ x, ...,\ x^{(n-1)}); \qquad x^{(k)}|_{t=0} = a_k, \qquad k = 1, 2, ...,\ n-1 \tag{4.4}$$

having $x(t)$ as its solution.

EXAMPLE 3. *(proving that the Conjecture above is false). The component $x(t)$ of the solution*

$$\begin{aligned} x(t) &= \frac{e^t - 1}{t} \\ y(t) &= \frac{1}{t} \end{aligned}$$

satisfying the IVP for a system of polynomial ODEs

$$\begin{aligned} x' &= x - xy + y, \qquad & x|_{t=1} = e - 1 \\ y' &= -y^2, \qquad & y|_{t=1} = 1 \end{aligned} \tag{4.5}$$

at all points t (except at $t = 0$) can not satisfy any explicit polynomial *ODE (4.4) at whichever point. At the point $t = 0$ this $x(t)$ also can not satisfy any regular* rational *ODE (4.2). (This is the main result of [3]).*

5. The general setting

Further on we assume that the right hand sides of the source system in the Conjecture are m nonzero polynomial forms of degree 2 in m variables (for simplicity written in dependent variables $x,\ y,\ z$)

$$\begin{aligned} x' &= F(t,\ x,\ y,\ z), \qquad & x|_{t=0} = a \\ y' &= G(t,\ x,\ y,\ z), \qquad & y|_{t=0} = b \\ z' &= H(t,\ x,\ y,\ z), \qquad & z|_{t=0} = c \end{aligned} \tag{5.1}$$

and we are looking for an equivalent $n-$order ODE say for x.

We can also assume that the solution $x(t)$ itself is not polynomial so that infinitely many $x^{(n)}|_{t=0} \neq 0$. (If $x(t)$ is a polynomial, it is obviously a solution of some polynomial $n-$order ODE, regular indeed as the Conjecture states).

Repeatedly differentiating the equation for x, the following infinite sequence of equations may be obtained:

$$
\begin{aligned}
x' &= F_1(t,\ x,\ y,\ z) = F(t,\ x,\ y,\ z) \\
x'' &= F_2(t,\ x,\ y,\ z) = \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial x}x' + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial z}z' = \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial x}F + \frac{\partial F_1}{\partial y}G + \frac{\partial F_1}{\partial z}H \\
&\quad .... \\
&\quad ..... \\
x^{(n+1)} &= F_{n+1}(t,\ x,\ y,\ z) = \\
&= \frac{\partial F_n}{\partial t} + \frac{\partial F_n}{\partial x}x' + \frac{\partial F_n}{\partial y}y' + \frac{\partial F_n}{\partial z}z' = \frac{\partial F_n}{\partial t} + \frac{\partial F_n}{\partial x}F + \frac{\partial F_n}{\partial y}G + \frac{\partial F_n}{\partial z}H \\
&\quad ....
\end{aligned} \tag{5.2}
$$

No equation $F_n(t,\ x,\ y,\ z)$ in this sequence may be a zero polynomial (otherwise all the subsequent $F_N(t,\ x,\ y,\ z)$, $N > n$, would be zero polynomials too, making the solution $x(t)$ a polynomial).

So all $F_n(t,\ x,\ y,\ z)$ are non-zero forms.

Given the fact that the operators $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ or $\frac{\partial}{\partial z}$ lower the degree of F_n by 1, while the degree of factors $F,\ G,\ H$ is 2, the degree of a form F_{n+1} is higher than that of F_n by 1. As the degree of F_1 is 2, therefore the degree of F_n is $n+1$.

The form $F_n(t,\ x,\ y,\ z)$ in the sequence (5.2) is expressed recursively via partial derivatives of the previous form $F_{n-1}(t,\ x,\ y,\ z)$ only (so that the recursive depth is 1).

Another format of the sequence of equations for $x^{(n)}$ may be obtained if we assume that forms $F,\ G,\ H$ are in squares only like (4.3). Then

$$
\begin{aligned}
x^{(n+1)} &= F_{n+1}(x,\ y,\ z) = \\
&= \sum_{i=0}^{n} C_n^i (a_{11}x^{(i)}x^{(n-i)} + a_{12}y^{(i)}y^{(n-i)} + a_{13}z^{(i)}z^{(n-i)}) = \\
&= \sum_{i=0}^{n} C_n^i (a_{11}F_iF_{n-i} + a_{12}G_iG_{n-i} + a_{13}H_iH_{n-i})
\end{aligned} \tag{5.3}
$$

where polynomials

$$
\begin{aligned}
G_{n+1} &= \sum_{i=0}^{n} C_n^i (a_{21}F_iF_{n-i} + a_{22}G_iG_{n-i} + a_{23}H_iH_{n-i}) \\
H_{n+1} &= \sum_{i=0}^{n} C_n^i (a_{31}F_iF_{n-i} + a_{32}G_iG_{n-i} + a_{33}H_iH_{n-i}).
\end{aligned}
$$

The form $F_{n+1}(x,\ y,\ z)$ in the sequence (5.3) is expressed recursively via the all previous forms F_i , G_j , H_k (avoiding partial differentiation) so that the recursive depth is n. Although for $i = 1$ forms $F_i(x,\ y,\ z)$, $G_i(x,\ y,\ z)$, $H_i(x,\ y,\ z)$ are particularly simple being in squares only, for $i > 1$ they lose this simplicity:

$$\begin{aligned}
F_1 &= a_{11}x^2 + a_{12}y^2 + a_{13}z^2 \\
F_2 &= 2a_{11}xx' + 2a_{12}yy' + 2a_{13}zz' = \\
&= 2a_{11}x(a_{11}x^2 + a_{12}y^2 + a_{13}z^2) + 2a_{12}y(a_{21}x^2 + a_{22}y^2 + a_{23}z^2) + \\
&\quad +2a_{13}z(a_{31}x^2 + a_{32}y^2 + a_{33}z^2) \\
&= 2a_{11}^2x^3 + 2a_{11}a_{12}xy^2 + 2a_{11}a_{13}xz^2 + 2a_{12}a_{21}x^2y + 2a_{12}a_{22}y^3 + 2a_{12}a_{23}yz^2 + ...
\end{aligned}$$

turning into general $i+1$ degree forms with all monomials present.

The infinite sequence of polynomial equations (5.2) or (5.3) is a starting point for various attempts to convert these explicit systems of ODEs into one implicit polynomial ODE of an order n, and to find a proof of the Conjecture.

6. Transformation into one $n-$order implicit polynomial ODE

The most general (albeit impractical) approach of obtaining one implicit polynomial ODE in $t,\ x,\ x', ...,\ x^{(n)}$ from (5.2) is elimination of "foreign" variables $y,\ z$ (in case of $m=3$) from that infinite system by means of resultants, as discussed in [2]. Formally any three equations (say numbered $\alpha < \beta < \gamma$ in the case of $m=3$) may be picked from system (5.2) for elimination of $y,\ z$ in order to obtain an implicit polynomial ODE

$$P_{\alpha\beta\gamma}(t,\ x,\ x^{(\alpha)},\ x^{(\beta)},\ x^{(\gamma)}) = P_{\alpha\beta\gamma}(T,\ X_0,\ X_\alpha,\ X_\beta,\ X_\gamma) = 0 \tag{6.1}$$

with the leading derivative $x^{(\gamma)} = X_\gamma$. However:

- We do not know whether the elimination process necessarily ends up with a nonzero polynomial, i.e. whether a nonzero polynomial equation (6.1) emerges in any elimination process at least for one triplet $\alpha,\ \beta,\ \gamma$.
- We do not know whether there exists at least one pair of equations (numbered $i,\ j$) in the system (5.2) *invertible* in $y,\ z$ at the given point, i.e. whether a Jacobian

$$J|_{t=0} = \left\| \begin{matrix} \dfrac{\partial F_i}{\partial y} & \dfrac{\partial F_i}{\partial z} \\ \dfrac{\partial F_j}{\partial y} & \dfrac{\partial F_j}{\partial z} \end{matrix} \right\|_{t=0} \neq 0. \tag{6.2}$$

  If no such pair of equations exists, the infinite columns $\left\| \dfrac{\partial F_i}{\partial y} \right\|_{t=0}$ and $\left\| \dfrac{\partial F_i}{\partial z} \right\|_{t=0}$ $(i = 1,\ 2, ...)$ in (5.2) are linearly dependent, (yet we do not know what to make of it).
- Even if (6.1) is a nonzero polynomial, we do not know whether

$$\left. \frac{\partial P_{\alpha\beta\gamma}}{\partial X_\gamma} \right|_{t=0} \neq 0.$$

Of those three questions, the answer is available only for the first one.

6.1. Transformation into a nonzero implicit polynomial ODE is possible. A proof of a possibility to transform a system of ODEs into an implicit polynomial ODE (6.4) below will be established promptly in Theorem 3.

The proof (whose idea belongs to Harley Flanders[1]) capitalizes on the fact from the combinatorics that the number $\pi(n)$ of partitions of n grows faster than the number of $n-$degree monomials in r variables (for any fixed r).

LEMMA 1. *The number C_{n+r-1}^{r-1} of all monomials comprising forms of degree n in r variables, and the number $\pi(n)$ of partitions of n satisfy the inequalities*

$$C_{n+r-1}^{r-1} < (n+r-1)^{r-1} < 2^{\sqrt{n}} < \pi(n) \tag{6.3}$$

beginning with a big enough $n > n_r$.

PROOF. The inequality $2^{\sqrt{n}} < \pi(n)$ is known say from [6], so we must prove that

$$(n+r-1)^{r-1} < 2^{\sqrt{n}}, \quad \text{or} \quad (n+r-1)^{\frac{r-1}{\sqrt{n}}} < 2,$$

which is equivalent to proving that

$$f(n) = \frac{(r-1)\ln(n+r-1)}{\sqrt{n}} < \ln 2.$$

Obtain

$$f'(n) = (r-1)\frac{\dfrac{\sqrt{n}}{n+r-1} - \dfrac{\ln(n+r-1)}{2\sqrt{n}}}{n} = (r-1)\frac{2n - (n+r-1)\ln(n+r-1)}{2(n+r-1)n\sqrt{n}}.$$

Beginning from big enough n, a linear function $2n < (n+r-1)\ln(n+r-1)$, so that $f'(n) < 0$ and $f(n)$ decreases. Moreover $\lim_{n\to\infty} \dfrac{(r-1)\ln(n+r-1)}{\sqrt{n}} = 0$ for any $r > 1$. Therefore $f(n) < \ln 2$ when $n > n_r$ for a big enough n_r. □

EXAMPLE 4.

$$r = 2,\ n_r = 4 : C_{4+2-1}^{2-1} = 5 = \pi(4) = 5,\ \textit{but}\ C_{5+2-1}^{2-1} = 6 < \pi(5) = 7.$$
$$r = 3,\ n_r = 13 : C_{13+3-1}^{3-1} = 105 > \pi(13) = 101,\ \textit{but}\ C_{14+3-1}^{3-1} = 120 < \pi(14) = 135.$$

THEOREM 3. *A system of explicit polynomial 1st order ODEs (5.1) is transformable into one implicit polynomial $n-$order ODE*

$$P(T,\ X_0, ...,\ X_{n-1}) = P(t,\ x, ...,\ x^{(n-1)}) = 0. \tag{6.4}$$

PROOF. Denote the forms of the sequence (5.2) as $P_k = F_{k-1}(t,\ x,\ y,\ z)$, $k = 2,\ 3,\ 4, ...$, assuming that P_1 is an arbitrary linear form with constants $A,\ B$: $P_1 = At + Bx$. This new notation is introduced in order that polynomials P_k be $k-$degree forms (unlike F_k which are $k-1$ degree forms).

Consider a sequence of sets $S_n,\ n = 1,\ 2,\ 3, ...$ each representing partitions of n

$$S_n = \{\ (\alpha_1, ..., \alpha_n) \mid \alpha_1 + 2\alpha_2 + ... + n\alpha_n = n\}$$

and special products $P_1^{\alpha_1} P_2^{\alpha_2} ... P_n^{\alpha_n}$, keeping in mind the one-to-one mapping

$$(\alpha_1, ..., \alpha_n) \Longleftrightarrow P_1^{\alpha_1} P_2^{\alpha_2} ... P_n^{\alpha_n}.$$

[1]From the private communications in 2004

Observe, that after completion of all the operations, every product $P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}$ becomes an $n-$degree form in $t,\ x,\ y,\ z$. According to the Lemma, for any given number of variables r there is such a number n_r that for any $n > n_r$ the number $\pi(n)$ of different partitions $(\alpha_1, ..., \alpha_n)$ exceeds the number C_{n+r-1}^{r-1} of all $n-$degree monomials $\{t^\alpha x^\beta y^\gamma z^\delta\}$ in r variables. Every product $P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}$ is an $n-$degree form comprised of the monomials $\{t^\alpha x^\beta y^\gamma z^\delta\}$. Therefore there exist a linear combination with nonzero coefficients $a_{\alpha_1,...,\alpha_n}$ such that

$$P = \sum_{(\alpha_1,...,\alpha_n)} a_{\alpha_1,...,\alpha_n} P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n} = 0$$

corresponding to the polynomial ODE

$$P(t,\ x,\ x',...,\ x^{(n-1)}) = \sum_{(\alpha_1,...,\alpha_n)} a_{\alpha_1,...,\alpha_n} (AT + BX_0)^{\alpha_1} X_1^{\alpha_2}...X_{n-1}^{\alpha_n} = 0. \tag{6.5}$$

□

Impractical just like the method of elimination via resultants, this Theorem too does not offer a feasible method for obtaining the ODE (6.5). However, unlike the resultants, the Theorem does guarantee that the desired polynomial ODE exists by producing a nonzero polynomial.

Remark 3. *Depending on a choice of the particular linear independent products $P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}$ and constants $A,\ B$, formally there may be infinitely many polynomial equations (6.5), yet for neither of them we know whether*

$$\left.\frac{\partial P}{\partial X_{n-1}}\right|_{t=0} \neq 0.$$

Remark 4. *If one ODE (6.5) is found, infinitely many of them may be obtained by differentiation $\left(\frac{d}{dt}\right)^N$ of (6.5)*

$$P(t,\ x,\ x',...,\ x^{(n-1)}) = 0$$

$$x^{(n)}\frac{\partial P}{\partial X_{n-1}} + Q_0(t,\ x,\ x',...,\ x^{(n-1)}) = 0 \tag{6.6}$$

$$x^{(n+1)}\frac{\partial P}{\partial X_{n-1}} + Q_1(t,\ x,\ x',...,\ x^{(n)}) = 0$$

$$...$$

$$x^{(n+N)}\frac{\partial P}{\partial X_{n-1}} + Q_N(t,\ x,\ x',...,\ x^{(n+N-1)}) = 0 \tag{6.7}$$

$$...$$

where Q_N $(N = 0,\ 1,\ ....)$ denote some polynomials in the specified variables. Unlike (6.5), all of these equations are linear *in the leading derivative $x^{(n+N)}$, and they all have the same critical factor $\frac{\partial P}{\partial X_{n-1}} = \frac{\partial P(T,\ X_0,\ X_1, ...,\ X_{n-1})}{\partial X_{n-1}}$ in all equations. Therefore if it happens that $\left.\frac{\partial P}{\partial X_{n-1}}\right|_{t=0} = 0$ in one equation (6.6) making it singular at this point, all the other ODEs (6.7) are singular too.*

Remark 5. *For a given n, the rank of the linear space $\mathcal{F}_r^n$ of all $n-$degree monomials in r variables is C_{n+r-1}^{r-1}. All $n-$degree forms generated by the products $\{P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}\}$ comprise a subspace $\mathbf{P}_n = Span\{P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}\}$ of monomials $\{t^\alpha x^\beta y^\gamma z^\delta\} \subset \mathcal{F}_r^n$ (albeit the number $\pi(n)$ of products $\{P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}\}$ exceeds C_{n+r-1}^{r-1}). Therefore $Rank(\mathbf{P}_n) = K_n \leq Rank(\mathcal{F}_r^n) = C_{n+r-1}^{r-1}$.*

We do not know which subset of K_n products $\{P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}\}$ comprises the basis of the subspace $\mathbf{P}_n$.

Note that α_n may take only values 1 or 0. At that, if $\alpha_n = 1$, then $\alpha_1 = \alpha_2 = ...\alpha_{n-1} = 0$. And if $\alpha_n = 1$, then $\alpha_1 + 2\alpha_2 + ... + (n-1)\alpha_{n-1} = n$.

Is it possible that the subset $\{P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}\} \setminus \{P_n\}$ (i.e. a subset $\{P_1^{\alpha_1}P_2^{\alpha_2}...P_{n-1}^{\alpha_{n-1}}\}$ where $\alpha_n = 0$) has the same span as $Span\{P_1^{\alpha_1}P_2^{\alpha_2}...P_n^{\alpha_n}\}$ for a big enough n? If it were possible, then $P_n \in Span\{P_1^{\alpha_1}P_2^{\alpha_2}...P_{n-1}^{\alpha_{n-1}}\}$ meaning that P_n is a linear combination of some products $P_1^{\alpha_1}P_2^{\alpha_2}...P_{n-1}^{\alpha_{n-1}}$ generating an explicit polynomial ODE

$$X_{n+1} = P_1(T,\ X,\ X', ...,\ X_n). \tag{6.8}$$

Yet we *must not* look for such an outcome, as the Example 3 shows.

However if we found a product $P_1^{\alpha_1}...P_k^{\alpha_k}P_{k+1}$ (rather than stand alone P_n) such that

$$P_1^{\alpha_1}...P_k^{\alpha_k}P_{k+1} \in Span\{P_1^{\alpha_1}P_2^{\alpha_2}...P_k^{\alpha_k}\}, \quad k < n,$$

it could deliver the desired non-singular ODE (providing that $|P_1^{\alpha_1}...P_k^{\alpha_k}|_{t=0} \neq 0$). Unfortunately, the structure of forms $P_1^{\alpha_1}P_2^{\alpha_2}...P_k^{\alpha_k}$ is too complicate for studying their basis.

6.2. Elimination of parameters not necessarily delivers a regular polynomial equation. In this subsection let us ignore for a while that all X_n in the polynomial equations $X_n = F_n(T,\ X,\ Y,\ Z)$ of the sequence (5.2) stand for derivatives of the function $x(t)$. Instead, consider the equations (5.2) as a parametric polynomial representation of a manifold (with the internal coordinates $(T,\ X,\ Y,\ Z)$) into a space with coordinates $(X_1,\ X_2, ...)$. Indeed, every point $(T,\ X,\ Y,\ Z)$ is regular because all the parametric equations are polynomial.

Now we want to transform this parametric representation of the manyfold into an implicit polynomial representation $P(T,\ X_1,\ X_2,\ X_3) = 0$. Is it always possible to find a polynomial P which is regular in a particular variable X_i (meaning $\left.\dfrac{\partial P}{\partial X_i}\right|_{t=t_0} \neq 0$) at a given point $(t_0,\ x_0,\ y_0,\ z_0) \longmapsto (T_0,\ X_1^0,\ X_2^0,\ X_3^0)$? The answer is "No" - even for one-dimensional manifold, i.e. for a curve.

Example 5. *A polynomial parametric curve*

$$\begin{aligned} x &= t^2 \\ y &= t \end{aligned}$$

(defining the positive branch of an algebraic curve $y = \sqrt{x}$) is regular for all t and particularly at $t = 0$. Yet the corresponding implicit polynomial equation $P = y^2 - x = 0$ (and any other polynomial equation satisfied by this $y(x)$) must be singular in y at the point $(x = 0,\ y = 0)$, i.e. $\left.\dfrac{\partial P}{\partial y}\right|_{x=0} = 0$, because this is a point of branch singularity of $y(x)$, so that the condition of existence and uniqueness of the solution in y at this point must be violated.

Example 6. *A polynomial parametric curve*

$$x = 1 - t^2$$
$$y = (1 - t^2)t$$

(defining the positive branch of an algebraic curve[2] $y = +x\sqrt{1-x}$*) is regular everywhere including at* $t = 1$*. Yet the corresponding implicit polynomial equation* $P = y^2 - x^2(1-x) = 0$ *is singular, and any other polynomial equation satisfied by this* $y(x)$ *must be singular in* y *at the point* $x = y = 0$ *(i.e.* $\left.\frac{\partial P}{\partial y}\right|_{x=0} = 0$*) because the algebraic curve* $y(x)$ *belongs to the algebraic pair* $y_{1,2}(x) = \pm x\sqrt{1-x}$*. The singularity point of* $y_{1,2}(x)$ *is* $x = 1$*. At the point* $x = y = 0$ *both branches are regular, but intersect each other, therefore any polynomial equation satisfied by such* $y(x)$ *must violate the condition of existence and uniqueness of the solution in* y *at the point* $x = y = 0$*.*

The function $y(x)$ at the point $x = y = 0$ cannot satisfy any implicit polynomial equation, yet it does satisfy:

- A regular algebraic equation $y = +x\sqrt{1-x}$;
- A regular IVP for a polynomial system of ODEs

$$y' = z - \frac{x}{2z}, \qquad y|_{x=0} = 0$$
$$z' = -\frac{1}{2z}, \qquad z|_{x=0} = 1, \qquad (z = \sqrt{1-x});$$

- A regular IVP for a polynomial ODE

$$y'' = -y'\frac{4-3x}{2(2-3x)(1-x)}, \qquad y|_{x=0} = 0, \qquad y'|_{x=0} = 1$$

Conclusion 1. *If the Conjecture is false, there must be an example of a polynomial system (5.2) and such a point* t*, at which any polynomial ODE (6.4) satisfied by* $x(t)$ *must be singular (in the leading derivative). For example, if the system (4.5) in Example 3 (or another such system) were valid also at* $t = 0$*, then the* $x(t)$ *would refute the Conjecture.*

Conclusion 2. *If the Conjecture is true, its proof must take into consideration the fact, that the variables* X_n *in the equations (5.2) stand for the derivatives of the solution* $x(t)$*, so that* $F_n(T,\ X,\ Y,\ Z)$ *are not arbitrary polynomials, but the polynomials satisfying the recursive formulas (5.2) or (5.3).*

Conclusion 3. *Unlike the parametric representation in the Examples above (comprised of two equations only), the sequences (5.2) or (5.3) contain infinite number of equations implementing the mapping*

$$(x(t),\ y(t),\ z(t)) \mapsto (x(t),\ x'(t),\ x''(t), ...,\ x^{(n)}(t), ...).$$

Therefore the sequences offer multiple choices how to select a triplet of equations from (5.2) for elimination of y *and* z*. We are looking for such a finite subspace of the infinite space* $(X_0,\ X_1,\ X_2, ...)$ *that the projection of the curve*

$$(x(t),\ x'(t),\ x''(t), ...,\ x^{(n)}(t), ...) \in (X_0,\ X_1,\ X_2, ...)$$

into $(X,\ Y,\ Z)$ *satisfy a regular polynomial at a given point* t*.*

[2] This is the so called "cubic with double point" - courtesy of George Bergman

7. Families of polynomial ODEs having the same solution

If one polynomial ODE (6.5) satisfied by a given holomorphic solution $x(t)$ exists, there exists also an *infinite family* of polynomial ODEs [3] satisfied by $x(t)$. Here the holomorphic solution is considered defined by its Taylor coefficients or derivatives $x^{(m)}|_{t=0} = b_m, \quad m = 0,\ 1,\ 2, ...$, while the ODEs of the family satisfied by $x(t)$ are sought as polynomial ODEs

$$P(t,\ x,\ x', ...,\ x^{(m)}) = P(T,\ X_0, ...,\ X_m) = \sum_{k=1}^{q} b_k T^{\alpha_k} X^{\beta_k} X_1^{\gamma_k} ... X_m^{\omega_k} = 0 \tag{7.1}$$

with unknown coefficients b_k. These coefficients satisfy an infinite algebraic system

$$\sum_{k=1}^{q} b_k M_{nk} = 0, \qquad n = 0,\ 1,\ 2, ... \tag{7.2}$$

obtained by differentiating the equation (7.1), where

$$M_{nk} = \left(\frac{d}{dt}\right)^n (T^{\alpha_k} X^{\beta_k} X_1^{\gamma_k} ... X_m^{\omega_k})_{t=0},$$

i.e. M_{nk} stands for $n-$th derivative of $k-$th monomial in (7.1). The linear algebraic system (7.2) is an infinite over-defined yet consistent system, whose one nonzero solution (a set of b_k) is known.

Now we can pose the following Problem.

Problem 1. *If the polynomial ODE (7.1) for one particular set of coefficients b_k happened to be singular so that $\left.\frac{\partial P}{\partial X_m}\right|_{t=0} = 0$: Is it possible to find another polynomial ODE in the infinite family which is regular at this point (or is it possible to widen the family by increasing the degree of the polynomial P and the number q of monomials in it)?*

Prior to consideration of the Problem, the following must be noted:

(1) Generally, not for any holomorphic function $x(t)$ a polynomial ODE satisfied by $x(t)$ and regular at $t = 0$ necessarily exists. For example, the function $x(t) = \frac{e^t - 1}{t}$ defined by its analytical element $x^{(k)}|_{t=0} = \frac{1}{k+1}$ at $t = 0 \quad (k = 0,\ 1, ...)$ can satisfy no polynomial ODE regular at $t = 0$ [3]. Therefore we must not expect that the regularization of a given polynomial ODE is possible for any arbitrary analytical element.

(2) The particularity of our setting is that the analytical element $x^{(k)}|_{t=0} = a_k \quad (k = 0,\ 1, ...)$ is not arbitrary: These are the values obtainable by evaluation of the special sequence (5.2). It is suspected (if the Conjecture is true) that the sequence (5.2) can not generate the derivatives like $x^{(k)}|_{t=0} = \frac{1}{k+1}$.

(3) Moreover, if we substitute the forms $X_k = F_k(t,\ x,\ y,\ z)$ of the sequence (5.2) into any polynomial $P(T,\ X_0, ...,\ X_m)$ of the family (7.1), those F_k will satisfy P turning it into a zero polynomial in $T,\ X,\ Y,\ Z$:

$$P(T,\ X,\ F_1(T,\ X,\ Y,\ Z), ...,\ F_m(T,\ X,\ Y,\ Z)) \equiv 0$$

(because the polynomial P was obtained such a way that F_k satisfy P).

For further consideration it makes sense to re-write the polynomial ODE (7.1) in the form of the Taylor polynomial at a point $(t_0,\ a_0,\ a_1,\ a_2, ...)$

$$0 = P = B(T - t_0) + A_0(X_0 - a_0) + ... + A_m(X_m - a_n)+ \tag{7.3}$$
$$+C(T - t_0)^2 + C_0(T - t_0)(X_0 - a_0) + C_1(T - t_0)(X_1 - a_1) + ...$$
$$+A_{00}(X_0 - a_0)^2 + A_{01}(X_0 - a_0)(X_1 - a_1) + A_{02}(X_0 - a_0)(X_2 - a_2) + ...$$

with unknown coefficients

$$B,\ C,\ C_0, ...,\ A_0, ...,\ A_m,\ A_{00},\ A_{01}, ...$$

The variables $X_0, ...,\ X_m$ in the source equation (7.3) and further $X_{m+1},\ X_{m+2}, ...$ after differentiation of (7.3) take the values a_k obtained by evaluation of the sequence (5.2) as emphasized in items (2, 3).

The advantage of the Taylor format (7.3) is in that the condition of the regularity of ODE (7.1) translates into the simplest verifiable form

$$\left.\frac{\partial P}{\partial X_m}\right|_{t=t_0} = A_m \neq 0.$$

In other words we are looking for such a solution (a set of coefficients)

$$B,\ C,\ C_0, ...,\ A_0, ...,\ A_m,\ A_{00},\ A_{01}, ...,$$

that A_m be nonzero. The values of the evaluated elements of the matrix of the linear algebraic system (7.2) computed for $x^{(k)}|_{t=t_0} = a_k$ are shown in Table 3:

$\frac{d^n}{dt^n}$	B	A_0	A_1	A_2	...	A_m	C	C_0	C_1	...	A_{00}	A_{01}	...
0	0	0	0	0	...	0	0	0	0	0	0	0	...
1	1	a_1	a_2	a_3	...	a_{m+1}	0	0	0	0	0	0	...
2	0	a_2	a_3	a_4	...	a_{m+2}	2	a_1	a_2	...	a_1^2	a_1a_2	...
3	0	a_3	a_4	a_5	...	a_{m+3}	0	$2a_2$	$2a_3$	...	...	...	...
4	0	a_4	a_5	a_6	...	a_{m+4}	0	$3a_3$	$3a_4$	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...	...

Table 3. The matrix of the linear algebraic system (7.2) for the polynomial (7.3). The rows $(n = 0,\ 1,\ 2, ...)$ represent the evaluated factors at the unknown coefficients $B,\ A_0, ...,\ A_m, ...$after $n-$th differentiation.

In order that A_m be nonzero, it is sufficient to prove that A_m is a free unknown in the set of unknowns so that we can choose it arbitrarily. In order that A_m be free, the column corresponding to A_m must linearly depend on the other columns.

It *can not* be the case if for example $a_k = \dfrac{1}{k+1}$ as explained in item (1). However it possibly can for the values a_k generated by the sequence (5.2). For example, we can assume that a_k are generated by evaluation of the system (5.3) in squares only. Unfortunately, so far we were unable to take advantage of this particularity of values a_k.

8. Conclusion

The Conjecture remains unresolved. Perhaps it presents a serious challenge. It is important to find its solution because a few other unresolved issues depend on it. Those issues are the following.

- There are two approaches to the definition of elementariness: either for vector-functions via a system of 1st order regular ODEs, or for stand alone functions via one $n-$order regular ODE. Are both definitions equivalent?
- There exist isolated special points in solutions of elementary ODEs where a solution is holomorphic, yet it cannot satisfy any regular IVP for an $n-$order rational or polynomial ODE [3, 7]. If the Conjecture is true, such functions cannot satisfy also any regular IVP for rational or polynomial *systems* of ODEs. If so, those isolated special points in elementary functions are the points where elementariness of these functions is violated.
- The issue of System-to-One-ODE conversion is too classical to remain unattended and unresolved.

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E-mail address: galex@ski.org
URL: http://www.resonoelusono.com/gofen