A stumbling problem

Alexander Gofen (March, 2024)

This text explains what was an error and what turned into a stumbling problem in an attempt to resolve the Conjecture and close the gap in the unifying view theory [1]. The manuscript submitted in January 2023 was declined without reviewing and without indicating any errors, but later I figured out on my own an error in Lemma 1, which invalidated the entire proof of the Conjecture. Nevertheless, the manuscript appeared as a preliminary draft (preprint) here [1].

What happened to be a stumbling block is this special exceptional situation formulated below.

Consider an IVP for a polynomial system

$$\begin{aligned} x' &= P_1(t,\ x,\ y,\ z), \qquad x|_{t=t_0} = a, \\ y' &= Q_1(t,\ x,\ y,\ z), \qquad y|_{t=t_0} = b, \\ z' &= R_1(t,\ x,\ y,\ z), \qquad z|_{t=t_0} = c, \end{aligned} \tag{1}$$

having indeed a holomorphic solution: in particular $x(t)$ with all its derivatives $x^{(k)}: \ \ x^{(k)}|_{t=t_0} = a_k,\ k = 1,\ 2, ...$ The original Conjecture was this.

Conjecture 1 *There exists a rational ODE and the IVP for it*

$$x^{(n+1)} = \frac{p(t,\ x, ...,\ x^{(n)})}{q(t,\ x, ...,\ x^{(n)})}, \qquad x^{(k)}|_{t=t_0} = a_k$$

with the denominator $q(t,\ x, ...,\ x^{(n)})|_{t=t_0} \neq 0$ *having the same solution* $x(t)$.

In the attempt of its proof, we consider an infinite sequence of polynomial equations - the Fundamental Sequence[1] for $x(t)$

$x' = P_1(t,\ x,\ y,\ z)$	$y' = Q_1(t,\ x,\ y,\ z, ...)$	$z' = ...$	...
... $x^{(k)} = P_k(t,\ x,\ y,\ z)$ $x^{(k+1)} = P_{k+1}(t,\ x,\ y,\ z)$ ...			

(2)

defined by the following recursion:

$$\begin{aligned} P_{k+1}(t,\ x,\ y,\ z, ...) &= \frac{\partial P_k}{\partial t} + \frac{\partial P_k}{\partial x}x' + \frac{\partial P_k}{\partial y}y' + \frac{\partial P_k}{\partial z}z' \\ &= \frac{\partial P_k}{\partial t} + \frac{\partial P_k}{\partial x}P_1 + \frac{\partial P_k}{\partial y}Q_1 + \frac{\partial P_k}{\partial z}R_1. \end{aligned} \tag{3}$$

[1] The difference between the equation $x^{(k)} = P_k(t,\ x,\ y,\ z, ...)$ and the multi-variate formula Faa-diBruno for $x^{(k)}$ is that the Faa-diBruno formula contains monomials over derivatives $(x^{(i)})^{\alpha}(x^{(j)})^{\beta}...(y^{(k)})^{\gamma}...(z^{(l)})^{\delta}...$ instead of monomials over $x,\ y,\ z, ...$. Indeed, the Faa-diBruno formula by itself (without any ODEs (1)) cannot spell out $x^{(k)},\ y^{(i)},\ z^{(j)}, ...$ Here too, we do not have the finite formulas for polynomials $P_k(x,\ y,\ z):$ we have only recurrence (3) for them.

(The similar infinite sequences may be written down also for $y^{(k)},\ z^{(k)}, ...$ if we needed them). The recursion also may be written as

$$\begin{aligned} P_{k+1} &= \frac{d}{dt} P_k, \qquad \text{where} \\ \frac{d}{dt} &= \frac{\partial}{\partial t} + P_1 \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y} + R_1 \frac{\partial}{\partial z} \end{aligned}$$

so that the operator $\left(\frac{d}{dt}\right)^k$ would be a Faa di-Bruno-type cumbersome multivariate polynomial expression over $P_1,\ Q_1,\ R_1$, their partial derivatives, and over the operators $\frac{\partial^{\alpha+\beta+\gamma+\delta}}{\partial t^\alpha \partial x^\beta \partial y^\gamma \partial z^\delta}$ - if we needed such explicit formula for $\left(\frac{d}{dt}\right)^k$.

We want to eliminate unnecessary variables $y,\ z, ...$ in the Fundamental Sequence (2) from some of the equations which are invertible. In attempt to do so, we stumble into the following question.

Consider for example variable z. We may presume that all $\frac{\partial P_k}{\partial z}$ are non-zero polynomials meaning that z does occur in every P_k. Of those non-zero polynomials $\frac{\partial P_k}{\partial z}$ some, however, may have a zero value at the given point so that the respective k-equation (2) is not invertible in z at this point.

It can happen, however, that for some special initial point $(t_0,\ a,\ b,\ c)$ all values $\left.\frac{\partial P_k}{\partial z}\right|_{(t_0,\ a,\ b,\ c)} = 0$ so that the infinite column

$$\left(\frac{\partial P_k}{\partial z}\right)_{t=t_0}, \qquad k = 1,\ 2, ... \tag{4}$$

is a zero column.

How to eliminate z in this case? This is the stumbling problem at the moment.

Remark 1 *"To eliminate z" means to find a smaller IVP*

$$\begin{aligned} x' &= A_1(t,\ x,\ y_1), \qquad x|_{t=t_0} = a, \\ y_1' &= B_1(t,\ x,\ y_1), \qquad y_1|_{t=t_0} = b_1 \end{aligned} \tag{5}$$

not containing z and having the same solution $x(t)$, i.e. the same sequence of derivatives $x^{(k)}|_{t=t_0}$(while the component y_1 may differ from y). Here the functions A_1 and B_1 are algebraic regular at the given point.

Remark 2 *The sequence $\{P_k\}$ proper represents k-derivatives $\left(\frac{d}{dt}\right)^k$ of $x(t)$ - but we cannot say anything about the sequences $\left(\frac{\partial P_k}{\partial x}\right),\ \left(\frac{\partial P_k}{\partial y}\right),\ \left(\frac{\partial P_k}{\partial z}\right)$, $k = 1,\ 2,\ ....$*

Remark 3 *The fact that all* $\left.\dfrac{\partial P_k}{\partial z}\right|_{t=t_0} = 0$ *in (3) creates an illusion as though the factor* $R_1|_{t=t_0}$ *does not matter and may be arbitrarily changed in (1) not affecting the values of* $x^{(k)}|_{t=t_0}$. *However, any change in* R_1 *or in the value* $R_1|_{t=t_0}$ *propagates into all polynomials* P_k *also (because of (3)) thus changing the values* $x^{(k)}|_{t=t_0}$.

Remark 4 *While the original Conjecture is a statement of a general nature, this stumbling problem is more narrow and more special. If an example disproving the Conjecture exists, it must involve such a zero column (say for* z*) preventing elimination of* z *(because in cases when no zero column exists for a given system, the Conjecture is proven).*

Theorem 1 *If in the zero column (4) it happened that not only* $\left(\dfrac{\partial P_1}{\partial z}\right)_{t=t_0} = 0$, *but* $\dfrac{\partial P_1}{\partial z} \equiv 0$ *being zero polynomials in* z, *then all polynomials* P_k *and* $\dfrac{\partial P_k}{\partial z}$, $k = 1,\ 2, ...,$ *are polynomials only in* $(t,\ x,\ y)$ *so that* z *is automatically eliminated.*

Proof. The condition $\dfrac{\partial P_1}{\partial z} \equiv 0$ means that in

$$P_1 = p_0(t,\ x,\ y) + p_1(t,\ x,\ y)z + p_2(t,\ x,\ y)z^2 + ...$$

all $p_k,\ k > 0$, are zero polynomials so that $P_1 = p_0(t,\ x,\ y)$. Applying the mathematical induction, check it for P_{k+1} :

$$P_{k+1} = \frac{dP_k}{dt} = \frac{\partial P_k}{\partial t} + \frac{\partial P_k}{\partial x} P_1 + \frac{\partial P_k}{\partial y} Q_1$$

■

What is the special meaning of the zero column

Consider the *general* solution $x(t;\ t_0,\ a,\ b,\ c)$, $y(t;\ t_0,\ a,\ b,\ c)$, $z(t;\ t_0,\ a,\ b,\ c)$ of the system (1) re-writing this system as

$$\begin{aligned} \frac{\partial x}{\partial t} &= P_1(t,\ x,\ y,\ z), \quad x|_{t=t_0} = a, \\ \frac{\partial y}{\partial t} &= Q_1(t,\ x,\ y,\ z), \quad y|_{t=t_0} = b, \\ \frac{\partial z}{\partial t} &= R_1(t,\ x,\ y,\ z), \quad z|_{t=t_0} = c \end{aligned} \tag{6}$$

with understanding that

$$\left.\frac{\partial x}{\partial a}\right|_{t=t_0} = 1; \quad \left.\frac{\partial x}{\partial b}\right|_{t=t_0} = 0; \quad \left.\frac{\partial x}{\partial c}\right|_{t=t_0} = 0;$$
$$\left.\frac{\partial y}{\partial a}\right|_{t=t_0} = 0; \quad \left.\frac{\partial y}{\partial b}\right|_{t=t_0} = 1; \quad \left.\frac{\partial y}{\partial c}\right|_{t=t_0} = 0;$$
$$\left.\frac{\partial z}{\partial a}\right|_{t=t_0} = 0; \quad \left.\frac{\partial z}{\partial b}\right|_{t=t_0} = 0; \quad \left.\frac{\partial z}{\partial c}\right|_{t=t_0} = 1.$$

Since introduction of the Unifying View [2] it was specially emphasized, that if $x(t;\ a,\ b,\ c)$ is vector-elementary in t because the right-hand sides (6) are rational or polynomial, this very $x(t;\ a,\ b,\ c)$ is not necessarily elementary in a, in b, or in c so that $\frac{\partial x(t;\ a,\ b,\ c)}{\partial a}$, $\frac{\partial x(t;\ a,\ b,\ c)}{\partial b}$, and $\frac{\partial x(t;\ a,\ b,\ c)}{\partial c}$ may not be necessarily expressible via a system of ODEs with rational right-hand side $\mathbf{R}(t,\ x,\ y,\ z)$.

However, the following is true.

Theorem 2 *If the component $x(t;\ a,\ b,\ c)$ is elementary in t, its partial derivatives $\frac{\partial x(t;\ a,\ b,\ c)}{\partial a}$, $\frac{\partial x(t;\ a,\ b,\ c)}{\partial b}$ and $\frac{\partial x(t;\ a,\ b,\ c)}{\partial c}$ (as functions of t) are also elementary in t.*

Proof. Consider for example the function $\frac{\partial x(t;\ a,\ b,\ c)}{\partial c}$ and obtain its derivative in t remembering that $x,\ y,\ z$ (inside P_1) are functions of $t,\ a,\ b,\ c$:

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial x}{\partial c} &= \frac{\partial}{\partial c}\frac{\partial x}{\partial t} = \frac{\partial}{\partial c}P_1(t,\ x,\ y,\ z)\\ &= \frac{\partial P_1}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial P_1}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial P_1}{\partial z}\frac{\partial z}{\partial c}.\end{aligned}$$

The right-hand side is a polynomial in $t,\ x,\ y,\ z,\ \frac{\partial x}{\partial c},\ \frac{\partial y}{\partial c}$, and $\frac{\partial z}{\partial c}$. Similarly

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial y}{\partial c} &= \frac{\partial}{\partial c}\frac{\partial y}{\partial t} = \frac{\partial}{\partial c}Q_1(t,\ x,\ y,\ z)\\ &= \frac{\partial Q_1}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial Q_1}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial Q_1}{\partial z}\frac{\partial z}{\partial c}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial z}{\partial c} &= \frac{\partial}{\partial c}\frac{\partial z}{\partial t} = \frac{\partial}{\partial c}R_1(t,\ x,\ y,\ z)\\ &= \frac{\partial R_1}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial R_1}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial R_1}{\partial z}\frac{\partial z}{\partial c}\end{aligned}$$

Therefore, if we add three new unknown functions

$$u = \frac{\partial x}{\partial c},\ v = \frac{\partial y}{\partial c},\ \ w = \frac{\partial z}{\partial c}$$

to the system (1), we obtain a closed polynomial system in 6 functions $x,\ y,\ z,\ u,\ v,\ w$

$$\begin{aligned}
\frac{\partial x}{\partial t} &= P_1(t,\ x,\ y,\ z) \\
\frac{\partial y}{\partial t} &= Q_1(t,\ x,\ y,\ z) \\
\frac{\partial z}{\partial t} &= R_1(t,\ x,\ y,\ z) \\
\frac{\partial u}{\partial t} &= \frac{\partial P_1}{\partial x}u + \frac{\partial P_1}{\partial y}v + \frac{\partial P_1}{\partial z}w \\
\frac{\partial v}{\partial t} &= \frac{\partial Q_1}{\partial x}u + \frac{\partial Q_1}{\partial y}v + \frac{\partial Q_1}{\partial z}w \\
\frac{\partial w}{\partial t} &= \frac{\partial R_1}{\partial x}u + \frac{\partial R_1}{\partial y}v + \frac{\partial R_1}{\partial z}w
\end{aligned} \tag{7}$$

demonstrating that $u,\ v,$ and w are vector-elementary in t. ∎

Remark 5 *The Fundamental sequence written for* $u^{(k)} = \dfrac{\partial x^{(k)}}{\partial c}$ *looks similar to that for* x*:*

$$u^{(k)} \quad = \quad \begin{matrix} \ldots \\ \dfrac{\partial}{\partial c}P_k = \dfrac{\partial P_k}{\partial x}u + \dfrac{\partial P_k}{\partial y}v + \dfrac{\partial P_k}{\partial z}w \\ \ldots \end{matrix} \tag{8}$$

Remark 6 *If we integrate this expended system in* $x,\ y,\ z,\ u,\ v,\ w,$ *then* $\left|\dfrac{\partial x(t)}{\partial c}\right|,\ \left|\dfrac{\partial y(t)}{\partial c}\right|,$ *and* $\left|\dfrac{\partial z(t)}{\partial c}\right|$ *may be viewed as measures of dependency of the solution on the initial value c, (or the measure of instability in c) varying with t.*

Theorem 3 *If the infinite column (4) is zero-column so that all*

$$\left.\frac{\partial P_k(t,\ x,\ y,\ z)}{\partial z}\right|_{(t_0,\ a,\ b,\ c)} = 0, \quad k = 1,\ 2,\ ....,$$

then not only does $\dfrac{\partial x(t_0,\ a,\ b,\ c)}{\partial c} = 0$ *at* $t = t_0$ *(as always), but* $\dfrac{\partial x(t,\ a,\ b,\ c)}{\partial c} \equiv 0$ *for any t and*

$$\frac{\partial x^{(k)}(t,\ a,\ b,\ c)}{\partial c} \equiv 0, \quad k = 0,\ 1,\ 2...$$

for any t. The vice versa is also true.

Proof. Apply $\dfrac{\partial}{\partial c}$ to any of the equations (2) remembering that $x,\ y,\ z$ (arguments of P_1) are functions of $t,\ a,\ b,\ c$:

$$\frac{\partial}{\partial c}x^{(k)} = \frac{\partial P_k}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial P_k}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial P_k}{\partial z}\frac{\partial z}{\partial c}$$

and consider it at $t = t_0$:

$$\left(\frac{\partial}{\partial c}x^{(k)}\right)_{t=t_0} = \left(\frac{\partial P_k}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial P_k}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial P_k}{\partial z}\frac{\partial z}{\partial c}\right)_{t=t_0}. \tag{9}$$

Here $\left.\frac{\partial x}{\partial c}\right|_{t=t_0} = 0$, $\left.\frac{\partial y}{\partial c}\right|_{t=t_0} = 0$, and $\left.\frac{\partial z}{\partial c}\right|_{t=t_0} = 1$. Even though $\left.\frac{\partial z}{\partial c}\right|_{t=t_0} \neq 0$, the factor $\frac{\partial P_k}{\partial z}$ is a zero column by the condition of the Theorem. Therefore for all k

$$\left.\frac{\partial x^{(k)}}{\partial c}\right|_{t=t_0} = \left.u^{(k)}\right|_{t=t_0} = 0, \quad k = 1,\ 2,\ .... \tag{10}$$

meaning that $\frac{\partial x(t,\ a,\ b,\ c)}{\partial c} \equiv 0$ for all t at the fixed given values $a,\ b,\ c$ for which the zero column takes place.

The vice versa. Let $\frac{\partial x(t,\ a,\ b,\ c)}{\partial c} \equiv 0$ for all t at the point $(t,\ a,\ b,\ c)$. Then also

$$\left(\frac{\partial}{\partial t}\right)^k \frac{\partial x}{\partial c} = \frac{\partial x^{(k)}}{\partial c} = 0, \quad k = 0,\ 1,\ 2,\ ....$$

for all t at the point $(t,\ a,\ b,\ c)$, including at $t = t_0$ so that (10) holds. Now reconsider the formula (9). In it $\left.\frac{\partial x}{\partial c}\right|_{t=t_0} = \left.\frac{\partial y}{\partial c}\right|_{t=t_0} = 0$, while $\left.\frac{\partial z}{\partial c}\right|_{t=t_0} = 1$ so that it must be that all $\left.\frac{\partial P_k}{\partial z}\right|_{t=t_0} = 0$, $k = 1,\ 2,\ ...$ meaning that the z-column is a zero column. ■

Corollary 4 *In the case of a zero column, the* expanded *system (7) has algebraic integrals.*

Proof. First, it is $u(t) \equiv 0$. Then, also $u^{(k)} \equiv 0$. Then, considering (8)

$$\frac{\partial P_k(t,\ x,\ y,\ z)}{\partial y}v + \frac{\partial P_k(t,\ x,\ y,\ z)}{\partial z}w \equiv 0 \tag{11}$$

are algebraic integrals of the expanded system (7). A $t = t_0$ where $v(t_0) = 0$ and $w(t_0) = 1$, we have what we already know: the zero column in z. ■

Remark 7 *Beside the fact that* $\frac{\partial x(t,\ a_0,\ b_0,\ c_0)}{\partial c} \equiv 0$, *we do not know anything about* $\frac{\partial^2 x(t,\ a_0,\ b_0,\ c_0)}{\partial c^2}$ *or higher derivatives in c. If we write down a multivariate Taylor expansion at a point* $(t,\ a_0,\ b_0,\ c_0)$, $t \neq t_0$, *a coefficient at the linear term* $(c - c_0)$ *is zero. In terms of* ε *and* δ *this means that for any* $t_1 \neq t_0$ *there exists small* ε *and* δ *such that if* $|c - c_0| < \delta$, *for the respective solution* $x(t,\ a_0,\ b_0,\ c)$

$$|x(t_1,\ a_0,\ b_0,\ c) - x(t_1,\ a_0,\ b_0,\ c_0)| < \varepsilon.$$

This motivates the following Definition.

Definition 5 *The solution corresponding to such an initial point* $(t_0,\ a_0,\ b_0,\ c_0)$ *which makes a zero column in the Fundamental sequence (2) is called an exceptional solution.of the ODE for* x.

The special meaning of linearly dependent columns

In the previous section we considered the solution of the system as a general solution each component of which depended on t and the set of the initial values considered as parameters. That was a particular case of dependency of the solution - dependency on the special type of parameters (the initial values).

Now consider a solution-vector $(x(t,\ p),\ y(t,\ p),\ z(t,\ p))$ depending on a parameter p. As a function of two independent variables t and p, such a vector generally may satisfy quite different systems of ODEs: one in independent variables t, the other in p. We do not know what is that system of ODEs in p. We postulate that this solution-vector satisfies the earlier considered system (6) in t:

$$\begin{aligned}\frac{\partial x}{\partial t} &= P_1(t,\ x,\ y,\ z), \qquad x|_{t=t_0} = a,\\ \frac{\partial y}{\partial t} &= Q_1(t,\ x,\ y,\ z), \qquad y|_{t=t_0} = b,\\ \frac{\partial z}{\partial t} &= R_1(t,\ x,\ y,\ z), \qquad z|_{t=t_0} = c\end{aligned} \tag{12}$$

which *hides* the parameter p (i.e. it does not appear in the right-hand sides). We realize that while $(x(t,\ p),\ y(t,\ p),\ z(t,\ p))$ is elementary in t due to (12), we do not know any rational system of ODEs demonstrating elementariness of $x(t,\ p)$ in p, i.e. we do not know any rational system

$$\begin{aligned}\frac{\partial x}{\partial p} &= r(p,\ x,\ y, ...)\\ &\ \ \cdot\ \ \cdot\ \ \cdot\end{aligned}$$

satisfied by $x(t,\ p)$.

Denote $\dfrac{\partial x}{\partial p} = u(t,\ p)$, $\dfrac{\partial y}{\partial p} = v(t,\ p)$, $\dfrac{\partial z}{\partial p} = w(t,\ p)$. We are to show, that these $u,\ v,$ and w are elementary in t.

Theorem 6 *If the component* $x(t,\ p)$ *is elementary in* t*, its partial derivative* $u(t,\ p) = \dfrac{\partial x}{\partial p}$ *is also elementary in* t.

Proof. Applying $\dfrac{\partial}{\partial p}$ to the system (12) we get

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 x}{\partial t \partial p} = \frac{\partial}{\partial p} P_1(t,\ x,\ y,\ z)\\ &= \frac{\partial P_1}{\partial x} u + \frac{\partial P_1}{\partial y} v + \frac{\partial P_1}{\partial z} w.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial Q_1}{\partial x}u + \frac{\partial Q_1}{\partial y}v + \frac{\partial Q_1}{\partial z}w \\ \frac{\partial w}{\partial t} &= \frac{\partial R_1}{\partial x}u + \frac{\partial R_1}{\partial y}v + \frac{\partial R_1}{\partial z}w\end{aligned}$$

Here is a polynomial system of ODEs for $\frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t},\ \frac{\partial w}{\partial t}$ - an extension of (12)

$$\begin{aligned}\frac{\partial x}{\partial t} &= P_1(t,\ x,\ y,\ z) \\ \frac{\partial y}{\partial t} &= Q_1(t,\ x,\ y,\ z) \\ \frac{\partial z}{\partial t} &= R_1(t,\ x,\ y,\ z) \\ \frac{\partial u}{\partial t} &= \frac{\partial P_1}{\partial x}u + \frac{\partial P_1}{\partial y}v + \frac{\partial P_1}{\partial z}w \\ \frac{\partial v}{\partial t} &= \frac{\partial Q_1}{\partial x}u + \frac{\partial Q_1}{\partial y}v + \frac{\partial Q_1}{\partial z}w \\ \frac{\partial w}{\partial t} &= \frac{\partial R_1}{\partial x}u + \frac{\partial R_1}{\partial y}v + \frac{\partial R_1}{\partial z}w\end{aligned}$$

demonstrating elementariness in t of $\frac{\partial x}{\partial p} = u,\ \frac{\partial y}{\partial p} = v,\ \frac{\partial z}{\partial p} = w.$ ■

Let's assume that the infinite (numeric) columns $\left(\frac{\partial P_k}{\partial x},\ \frac{\partial P_k}{\partial y},\ \frac{\partial P_k}{\partial z}\right)_{t=t_0}$, $k = 1,\ 2,\ ...$ are linearly dependent with respective coefficients $\alpha,\ \beta,\ \gamma$ not all zeros so that

$$\left(\alpha\frac{\partial P_k}{\partial x} + \beta\frac{\partial P_k}{\partial y} + \gamma\frac{\partial P_k}{\partial z}\right)_{t=t_0} = 0, \quad k = 1,\ 2,\ ...$$

Set the initial values $u|_{t=t_0} = \alpha,\quad v|_{t=t_0} = \beta,\quad w|_{t=t_0} = \gamma$ so that

$$\left.\frac{\partial u}{\partial t}\right|_{t=t_0,\ a,\ b,\ c} = \left(\frac{\partial P_1}{\partial x}u + \frac{\partial P_1}{\partial y}v + \frac{\partial P_1}{\partial z}w\right)_{t=t_0,\ a,\ b,\ c} = 0.$$

Theorem 7 *If the infinite numeric columns* $\left(\frac{\partial P_k}{\partial x},\ \frac{\partial P_k}{\partial y},\ \frac{\partial P_k}{\partial z}\right)_{t=t_0,\ a,\ b,\ c}$ *are linearly dependant at the initial point* $t = t_0$*, then not only does* $\left.\frac{\partial u}{\partial t}\right|_{t=t_0,\ a,\ b,\ c} = 0$*, but* $\left.\frac{\partial u}{\partial t}\right|_{a,\ b,\ c} \equiv 0$ *for any* t *at the same initial point* $(a,\ b,\ c)$*.*

Proof. Just as before, apply $\frac{\partial}{\partial p}$ to the equations of the fundamental sequence (2)

$$\frac{\partial}{\partial p}\left(\frac{\partial x}{\partial t}\right)^k = \frac{\partial x^{(k)}}{\partial p} = \frac{\partial P_k}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial P_k}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial P_k}{\partial z}\frac{\partial z}{\partial p}$$

so that

$$\left.\frac{\partial x^{(k)}}{\partial p}\right|_{t=t_0} = \left(\frac{\partial P_k}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial P_k}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial P_k}{\partial z}\frac{\partial z}{\partial p}\right)_{t=t_0}$$

$$= \left(\frac{\partial P_k}{\partial x}\alpha + \frac{\partial P_k}{\partial y}\beta + \frac{\partial P_k}{\partial z}\gamma\right)_{t=t_0} = 0, \qquad k = 1,\ 2,\ ...$$

As $\left(\frac{\partial}{\partial t}\right)^k \left.\frac{\partial x}{\partial p}\right|_{t=t_0} = \left(\frac{\partial u}{\partial t}\right)^k_{t=t_0} = 0$ for all $k = 1,\ 2,\ ...,$ therefore $u \equiv \alpha$ also for any t at the same initial point $(a,\ b,\ c)$. ∎

Remark 8 *Though* $u \equiv \alpha$ *and*

$$\frac{\partial u}{\partial t} = \frac{\partial P_1}{\partial x}u + \frac{\partial P_1}{\partial y}v + \frac{\partial P_1}{\partial z}w \equiv 0,$$

the linear combination

$$\alpha\frac{\partial P_k}{\partial x} + \beta\frac{\partial P_k}{\partial y} + \gamma\frac{\partial P_k}{\partial z}$$

is zero only at $t = t_0$ *because only at this point* $u|_{t=t_0} = \alpha,\ \ v|_{t=t_0} = \beta,\ \ w|_{t=t_0} = \gamma$ *as they were set.*

We see that the fact of a zero column at a point and the fact of linearly dependent columns at the point leads to the similar identities for the parametric derivative $\frac{\partial x}{\partial p}$. "So what?!" - a question arises. How does it help to eliminate z?

Example 1

$$\begin{aligned} x' &= x + (x-y)z, & x(0) &= a \\ y' &= y + (x-y)z & y(0) &= a \\ z' &= R_1(t,\ x,\ y,\ z) & &\textit{whatever expression.} \end{aligned}$$

For these special initial values the solution of this system $x = y = ae^t$ *is exceptional because*

$$x^{(k+1)} = P_{k+1} = P_k + \sum_{i=0}^{k} C_k^i (x-y)^{(i)} z^{(k-i)}$$

and $\left.\frac{\partial P_k}{\partial z}\right|_{t=0} = 0$ *for all* k *because* $x \equiv y$ *is an integral of this IVP. Moreover, here not only does* $\left.\frac{\partial P_k}{\partial z}\right|_{t=0} = 0$*, but* $\frac{\partial P_k}{\partial z} \equiv 0$.

Let's consider three cases of the consequences of presence of a zero column for z.

1. Though the ODE for x formally contains the variable z, the vaiable z and the ODE for z are irrelevant for this particular solution $x(t)$. Such a remarakble case is illustrated by the Example above meaning the following. If we re-write the ODE for x and y as a polynomials in z

$$\begin{aligned} x' &= P_1(t,\ x,\ y,\ z) = p_0(t,\ x,\ y) + p_1(t,\ x,\ y)z + p_2(t,\ x,\ y)z^2 + ... \\ y' &= Q_1(t,\ x,\ y,\ z) = q_0(t,\ x,\ y) + q_1(t,\ x,\ y)z + q_2(t,\ x,\ y)z^2 + ... \end{aligned}$$

then for the given initial values all polynomial coefficients

$$p_j(t,\ x,\ y) \equiv 0 \text{ and } q_j(t,\ x,\ y) \equiv 0$$

all being integrals for such initial values. In this case z is automatically eliminated because actually

$$\begin{aligned} x' &= P_1(t,\ x,\ y,\ z) = p_0(t,\ x,\ y) \\ y' &= Q_1(t,\ x,\ y,\ z) = q_0(t,\ x,\ y). \end{aligned}$$

As to the general solution $x(t;\ t_0,\ a,\ b,\ c)$, here not only does

$$\frac{\partial x(t;\ t_0,\ a,\ b,\ c)}{\partial c} \equiv 0,$$

but

$$\frac{\partial x^m(t;\ t_0,\ a,\ b,\ c)}{\partial c^m} \equiv 0$$

for any m - the meaning of irrelevancy of variable z.

Moreover, not only does the solution $x(t)$ not depend on the value $z|_{t=0}$, but even the right-hand side of equation for z' has no effect on the $x(t)$ for these special initial values. Therefore, in this Example, in order to get rid of z obtaining a reduced system (5) it's enough to remove the zero polynomial (in z), namely $(x-y)z$ in both ODEs. However, in a general case of a zero column and the exceptional solution, we have no knowledge what to do in order to obtain the reduced system (5).

In the Examples below demonstrating linear dependency of the columns at a point or the zero column, elimination of z happens to be possible, however I do not know how to prove it (if this hypothesis is true).

Examples

Example 2 *Linearly dependent columns (zero Jacobian). Consider the IVP*

$$\begin{aligned} x' &= y + z; \quad x(0) = a \\ y' &= y^2; \quad y(0) = b \\ z' &= 2z^2; \quad z(0) = c \end{aligned}$$

whose solution is

$$\begin{aligned} x &= -\ln(1-tb) - \frac{1}{2}\ln(1-2ct) + a \\ y &= \frac{b}{1-bt}, \\ z &= \frac{c}{1-2ct}. \end{aligned}$$

The second and third ODEs are actually stand alone ODEs. We can write down their n-derivatives of the solutions

$$\begin{aligned} y^{(n)} &= n!y^{n+1} \\ z^{(n)} &= 2^n n! z^{n+1} \end{aligned}$$

and therefore we have expressions for P_n

$$x^{(n)} = P_n(x,\ y,\ z) = n!y^{n+1} + 2^n n! z^{n+1}$$

and

$$\frac{\partial P_n}{\partial y} = (n+1)!y^n; \qquad \frac{\partial P_n}{\partial z} = 2^n(n+1)!z^n.$$

The Jacobian J_{mn} *of lines* m *and* n, $n > m$ *is*

$$\begin{aligned} J_{mn} &= \begin{bmatrix} (n+1)!y^n & 2^n(n+1)!z^n \\ (m+1)!y^m & 2^m(m+1)!z^m \end{bmatrix} \\ &= 2^m(m+1)!z^m(n+1)!y^n - 2^n(n+1)!z^n(m+1)!y^m \\ &= 2^m(m+1)!(n+1)y^m z^m(y^{n-m} - (2z)^{n-m}). \end{aligned}$$

If the initial values are such that $b = 2c$, *all* $J_{mn}|_{t=0} = 0$ *meaning that columns* $\left.\frac{\partial P_n}{\partial y}\right|_{t=0}$ *and* $\left.\frac{\partial P_n}{\partial z}\right|_{t=0}$ *are linearly dependent when* $y|_{t=0} = b = 2c$, $z|_{t=0} = c$, *namely*

$$\begin{aligned} \left.\frac{\partial P_n}{\partial z}\right|_{t=0} &= 2^n(n+1)!c^n; \quad \left.\frac{\partial P_n}{\partial y}\right|_{t=0} = (n+1)!(2c)^n = 2^n(n+1)!c^n \\ \left.\frac{\partial P_n}{\partial z}\right|_{t=0} &= \left.\frac{\partial P_n}{\partial y}\right|_{t=0} \end{aligned}$$

so that $\alpha = 0$, $\beta = 1$, $\gamma = -1$ *in terms of Theorem 6. Now observe, that with such special initial values* $b = 2c$ *we can see that* y *and* z *are related:*

$$\begin{aligned} y &= \frac{b}{1-bt} = \frac{2c}{1-2ct} \\ z &= \frac{c}{1-2ct} \end{aligned}$$

i.e. $y \equiv 2z$, *being an integral of this IVP for these special initial values so that* z *can be eliminated.*

Example 3 *A zero column for particular initial values with nonzero polynomials. Consider the same IVP when* $b = 0, \quad c \neq 0$. *Now we see that* $\left.\dfrac{\partial P_n}{\partial y}\right|_{t=0} = 0$ *for all* n. *Observe again, that with these special initial values, the solution component* $y = const = 0$, *though* $\dfrac{\partial P_n}{\partial y}$ *is not a zero polynomial.*

Example 4 *All* $\dfrac{\partial P_n}{\partial z}$ *are zero polynomials. That is the case if* P_1 *and* Q_1 *in (1) do not contain* z *so that the subsystem in* $x, \ y$ *is self-contained.*

Example 5 *The nonzero column for any initial values. Consider an IVP*

$$\begin{aligned} x' &= x + y - xy, \qquad & x(1) = e - 1 \\ y' &= -y^2, & y(1) = 1 \end{aligned}$$

whose solution[2] *is an entire function* $x = \dfrac{e^t - 1}{t}, \quad x(0) = 1, \quad$ *(with* $y = \dfrac{1}{t}$ *having a singularity at* $t = 0$*). Then:*

$$P_1 = x + y - xy, \qquad \frac{\partial P_1}{\partial y} = 1 - x$$

Observe that $\left.\dfrac{\partial P_1}{\partial y}\right|_{t=1} = 2 - e \neq 0$ *so that at* $t = 1$ *the column* $\left.\dfrac{\partial P_k}{\partial y}\right|_{t=1}$ *cannot be zero column. For other values of* t, $\dfrac{\partial P_1}{\partial y}$ *may be zero only if* $x = 1$ *(with any* y*). However, the function* $x(t)$ *is such that* $x = 1$ *only at* $t = 0$ *which is inaccessible in this system. Therefore, the column* $\dfrac{\partial P_k}{\partial y}$ *cannot be zero column with any* t *for this system.*

1. The Gap in the Unifying View not yet Closed.
http://taylorcenter.org/Gofen/GapNotYetClosed.pdf
2. The Unifying view on ODEs and AD.

[2] This function $x(t)$ was proven to have violation of the scalar elementariness at $t = 0$.