

**The Ordinary Differential Equations and  
Automatic Differentiation  
Unified**

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*Complex Variable and Elliptic Equations*, Vol. 54, No. 9,  
September 2009, 825-854.

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1991 *Mathematics Subject Classification.* 34A09, 34A34, 34A12

*Key words and phrases.* ODE, AD, Removable singularities, Analytical continuation

ABSTRACT. Analytical continuation of a function represented with an arbitrary Taylor expansion at only one point is not constructive (in the sense further explained). What makes it constructive is an equation (algebraic or differential of certain class) defining this function and enabling its continuation by means of integration of the ODEs. Such functions comprise an important sub-class of holomorphic functions - *general elementary* functions widening the class of conventional elementary functions so that it becomes closed. In terms of the generalizing definition, the solutions of elementary ODEs are elementary too. In a frame of the Unifying view based on the general elementary functions, Automatic Differentiation merges with the theory of holomorphic ODE's. It is showed, that continuation of (general) elementary functions via integration of its ODEs not necessarily expands them into each and every point where these functions exist and are holomorphic. Some entire functions are suspects for being elementary everywhere except an isolated point: the point of their "removable" or "regular" singularity. Thus the unifying view uncovers a new meaning of the notion "removable singularity" as a new type of special point (which is rather "unmovable", being proper to a particular holomorphic function).

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## Preface

This survey attempts to gather and systematically present scattered facts related to Ordinary Differential Equations (ODEs) and computability of  $n$ -order derivatives in a frame of one theory.

Typically ODEs are considered for classes of real valued  $n$ -times differentiable functions. However all along this survey we are to deal with ODEs over a class of holomorphic functions only for the following reasons.

### 1. ODEs as generators of $n$ -order derivatives and Taylor expansions

A particularity of our approach is that we regard explicit systems of ODEs as a tool for computing  $n$ -order derivatives of the solution at points of the phase space, thus obtaining the Taylor expansions representing the holomorphic solution.

That implies that derivatives in this paper are in fact complex derivatives in the complex plane  $\mathbf{C}$ , (rather than derivatives along the real axis). However we may not necessarily mention explicitly the complexity of the functions we deal with. After all, operations over them in the complex plane  $\mathbf{C}$  formally look as though on the real axis.

### 2. From conventional to general elementary functions

In the 19th century and earlier, ideally mathematicians wished to obtain solutions in terms of the so called elementary functions. Those were a limited number of conventionally chosen well studied (and tabulated) functions such as trigonometric and their inverse, exponential, logarithm, rational functions, and finite superpositions constructed of them (Liouville).

Indeed, it was soon learned that the class of these conventional elementary functions is not closed even for operation of obtaining an antiderivative, not to mention more sophisticated ODEs. We are going to widen the concept of the elementary functions in such a way that the solutions of the general elementary ODEs always belong to the same class of general elementary functions too.

The fact of the 19th century that certain functions were tabulated, well studied, and targeted as the desired format of solutions, is no more relevant in the modern times. It makes more sense to conceptualize and generalize elementary functions on the basis of a certain fundamental property rather than an arbitrary convention.

Such a generalization of elementary functions was suggested by R. Moore in the 1960s (being further developed by A. Gofen). Moore defined general elementary functions as those that may be represented as solutions of explicit (nonlinear) ODEs whose right hand sides are rational in unknown functions and the variable of integration.

All the conventional elementary functions satisfy this definition, indeed. However, a much wider class of functions satisfies it too.

### 3. Terminology

All along this survey we deal only with the *general elementary* functions. Yet for the matter of brevity from now on we will call them here simply *elementary*. Elementary systems of ODEs mean system of ODEs whose right hand sides are general elementary functions. Elementariness of a (vector-) function means its property of being general elementary (which may be violated at certain points).

### 4. Closedness and fundamental transforms

As we will see, all ODEs with elementary right hand sides may be transformed to ODEs with rational and further to polynomial and quadratic right hand sides. Moreover, the widened class of elementary functions is *closed* in the sense that solutions of elementary ODEs are elementary too (and practically all ODEs appearing in applications are elementary indeed).

### 5. When analytical continuation is constructive

The fundamental fact about holomorphic functions is that its Taylor expansion at only one point (one analytic element) suffices for analytical continuation of this function into the entire domain of its existence. However this procedure is not constructive<sup>1</sup>. Say, we are given an arbitrary analytic element, a Taylor expansion with a non-zero convergence radius  $r$ , whose coefficients are defined via certain recursive formulas

$$u(t) = \sum_{k=0}^{\infty} a_k (t - t_0)^k.$$

In principle it may be expanded into another point  $t_1$  (providing that  $|t_1 - t_0| < r$ ), where it expansion would be

$$u(t) = \sum_{k=0}^{\infty} b_k (t - t_1)^k.$$

However we do not know how to constructively transform  $a_k$  into  $b_k$ , because it requires computation of infinite sums for which the method of convolution into finite expressions is generally unknown.

For example, if  $a_k = \frac{1}{k^k}$  at  $t_0 = 0$ , the series has an infinite convergence radius and defines an entire function  $u(t)$ , but we have no constructive way of obtaining its Taylor coefficients at points other than  $t_0 = 0$ , nor do we know any other representation.

Is it possible to obtain a more general expression for Taylor coefficients  $a_k(t_0) = f(k, t_0)$  generating them at any desired point  $t_0$  for  $u(t)$ ? The answer is yes, however the expression is the Taylor formula  $a_k = \frac{u^{(k)}(t_0)}{k!}$  for this very unknown function  $u(t)$  - rather a useless tautological statement for the purpose of continuation.

Fortunately, in physical and scientific applications the functions of interest are initially defined not by their expansions at single points, but rather as solutions of equations. If  $u(t)$  is defined or known to satisfy an algebraic or differential

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<sup>1</sup>With respect to analytical continuation, the term "constructive" applies to a process as efficient and accurate, as a process of numeric integration of ODEs. "Not constructive" means that no such process is available.

equation  $u' = F(u, t)$ , the equation enables computation of the Taylor coefficients at any (regular) point  $t_0$ , becoming a constructive tool for analytical continuation via integration.

However the equation is as much constructive<sup>2</sup> as our ability to obtain  $n$ -order derivatives of a composite function  $F(u, t)$ , which in a general case requires repeated application of the Faa di-Bruno formula of exponential complexity (further discussed in the Introduction).

## 6. When $n$ -order differentiation is constructive

Here is one more manifestation of the special role of elementary functions in ODEs. It is exactly the class of elementary functions, when  $n$ -order differentiation is possible in constructive way. By that, we mean a possibility of applying the optimized formulas for differentiation of expressions over operands (whose derivatives are considered already known).

The modern Taylor method and optimization of  $n$ -order differentiation emerged in the 1960s. We will use the term Automatic Differentiation (AD)<sup>3</sup> narrowly, referring only to this optimized  $n$ -order differentiation.

If ODEs and AD are considered as separate notions, the AD (the Taylor integration) is a numeric method (one of many), while the ODEs are objects to which the method is applied. On the contrary, we consider explicit ODEs as a tool of AD and of analytic continuation (even if only along the real axis), so that ODEs and the AD represent two sides of the same thing.

## 7. Summary

Here are seemingly unrelated concepts:

- Elementary functions;
- The class of elementary ODEs closed with regard to their solutions;
- The transforms of all elementary ODEs to the special formats, enabling...
- Optimized computability of  $n$ -order derivatives, enabling...
- The modern Taylor method, and analytic continuation;
- Special points unreachable via integration of ODEs - points of violation of elementariness.

They all however are interdependent and complementary to each other. An approach revealing the merger of these concepts is important enough by itself, so that we call it the *Unifying view on ODEs and AD* framed by the theory of holomorphic functions.

Chapter 1 introduces the definition of elementary functions. Chapter 2 presents the fundamental theorems, including the Main theorem establishing that the class of general elementary functions is closed.

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<sup>2</sup>With respect to  $n$ -order differentiation, the term "constructive" applies to algorithms of the polynomial complexity. Term "non-constructive" applies to those of exponential complexity like the Faa di-Bruno formula, or when no differentiation formula is available. In the latter case when nothing but the definition  $f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$  is available, its direct numerical implementation results in a catastrophic subtractive error.

<sup>3</sup>Generally the term Automatic Differentiation stands for modern techniques of converting code which computes a function into the code computing the  $n$ -order derivatives of the function - see the survey [Corliss, 2005]. However all along this paper AD is understood more narrowly as optimized formulas for  $n$ -order differentiation.

The chapter "Fundamental transforms" covers the conversions of all elementary ODEs to special formats such as the second degree polynomials.

In the final chapter we consider open issues such as the nature of the so called unremovable "removable" singularities as exceptional points, where holomorphic functions lost their elementariness, for example the function  $\frac{\sin z}{z}$  at the point  $z = 0$ .



## CHAPTER 1

# Introduction

Issues of computability of  $n$ -order derivatives and its optimization re-emerged in the 1960s thanks to Ramon Moore. These issues are contained in the vague term "Automatic Differentiation" understood here as the optimized process of obtaining  $n$ -order derivatives. Being a relatively modern topic, it however is based on the classical Taylor method for ODEs (B. Taylor, 1715) and formulas established in the early eighteenth century by Arbogast [in Leipnik, Pearce 2007].

In the frame of the Taylor method, an Initial Value Problem (IVP) for an explicit system of Ordinary Differential Equations (ODEs)

$$(1.1) \quad \{u'_k = f_k(u_1, \dots, u_m), \quad u_k|_{t=t_0} = a_k, \quad k = 1, \dots, m$$

is viewed as an explicit finite difference system for computing  $n$ -order derivatives (or the Taylor coefficients) of the solution, the recursive depth of the source equations (1.1) being 1.

Then the operator of differentiation should be applied repeatedly to both sides, increasing the recursive depth. It is possible in principle to obtain any  $n$ -order derivatives of the solution either as symbolic formulas of growing complexity, or as the numeric values. The latter is our goal farther on along this study.

The concept itself of constructive differentiation of the right hand sides  $f_k$  presumes a possibility of applying certain known formulas of differentiation. It is doable if all  $f_k$  are finite compositions over a limited list of functions whose derivatives are known. (Otherwise, for an arbitrary holomorphic right hand side  $f_k(u_1, \dots, u_m)$ , we would have no available formulas to apply but the definition of the derivative).

Traditionally this limited list of functions was comprised of *conventional* elementary functions. We are going to widen it so that:

- it will be defined not by a convention, but on the basis of a fundamental property;
- this class will be wider than a limited list of *conventional* elementary functions, and
- it will be closed in the sense that the solutions of elementary ODEs are elementary too.

### 1. Some technicalities of $n$ -order differentiation

Complexity of computing  $n$ -order derivatives of the solution of ODEs depends on complexity of the right sides on the one hand, and on the kind of differentiating algorithm applied to the particular ODEs.

For example, in the simplest case of linear systems with constant coefficients

$$\mathbf{u}' = A\mathbf{u} + \mathbf{b}$$

where  $A$  is a constant matrix, and  $\mathbf{b}$  is a constant vector, the  $n$ -order derivatives may be trivially obtained: either recursively  $\mathbf{u}^{(n+1)} = A\mathbf{u}^{(n)}$ , or even as the finite difference solution  $\mathbf{u}^{(n)} = A^n\mathbf{u} + A^{n-1}\mathbf{b}$ ,  $n = 1, 2, \dots$

However for nonlinear right hand sides the direct process of repeated differentiation generates explicit finite difference equations of growing recursive depth  $n$ :

$$u_k^{(n+1)} = [f_k(u_1, \dots, u_m)]^{(n)} = f_{kn}(u_1, \dots, u_m, \dots, u_1^{(n)}, \dots, u_m^{(n)}).$$

Generally  $n$ -order derivative  $[f(u_1, \dots, u_m)]^{(n)}$  is expressed via the multivariate formula of Faa di-Bruno [Leipnik, Pearce 2007]

(1.2)

$$[f(u_1, \dots, u_m)]^{(n)} = n! \sum_S \frac{\partial^{n_1+n_2+\dots+n_m}}{\partial u_1^{n_1} \partial u_2^{n_2} \dots \partial u_m^{n_m}} f(u_1, \dots, u_m) \prod_{i=1}^m \prod_{j=1}^n \frac{1}{n_{ij}!} \left( \frac{u_i^{(j)}}{j!} \right)^{n_{ij}}$$

where  $n_i = \sum_{j=1}^n n_{ij}$  and summation is performed over a set  $S$  of index matrixes

$$\|n_{ij}\|_{i,j=1}^{m,n}, n_{ij} \geq 0$$

$$S = \left\{ \|n_{ij}\| \left| \sum_{j=1}^n j \sum_{i=1}^m n_{ij} = n \right. \right\},$$

so that  $\sum_{i=1}^m n_i = \sum_{j=1}^n \sum_{i=1}^m n_{ij} \leq n$ .

For example:

$$\begin{aligned} [f(u, v)]' &= f'_u u' + f'_v v' \\ [f(u, v)]'' &= f''_{uu} (u')^2 + 2f''_{uv} u' v' + f''_{vv} (v')^2 + f'_u u'' + f'_v v'' \\ [f(u, v)]''' &= f'''_{uuu} (u')^3 + 3f'''_{uuv} (u')^2 v' + 3f'''_{uvv} u' (v')^2 + f'''_{vvv} (v')^3 + \\ &\quad + 3f''_{uu} u' u'' + 3f''_{vv} v' v'' + 3f''_{uv} u'' v' + 3f''_{uv} u' v'' + \\ &\quad + f'_u u''' + f'_v v''' \end{aligned}$$

For the single variable formula ( $m = 1$ ) the number of its terms is the number of solutions of the equation

$$\sum_{j=1}^n j n_j = n.$$

That is the number of partitions of  $n$ , known to grow exponentially. The number of terms in the multivariate ( $m > 1$ ) Faa di Bruno formula obviously exceeds that number for the single variable formula, growing even faster.

Moreover, in order to apply the Faa di Bruno formula for  $n$ -order differentiation of ODEs, we would need an algorithm of obtaining all partial derivatives  $\frac{\partial^{n_1+n_2+\dots+n_m}}{\partial u_1^{n_1} \partial u_2^{n_2} \dots \partial u_m^{n_m}} f(u_1, \dots, u_m)$  for *arbitrary* right hand sides. For example, function  $f$  in turn may break down into composition of several other functions, so that we might need even more complex multi-chained version of the Faa di-Bruno formula.

Fortunately, instead of dealing with arbitrary right hand sides, it is possible:

- First, to convert elementary (non-rational) ODEs to rational right hand sides (established in the Main Theorem later), and

- Second - to optimize  $n$ -order differentiation of rational expression so that the amount of the required computation becomes no more than  $O(n^2)$  operations.

The Faa di Bruno formula is indeed the most general form for  $n$ -order derivatives of the solution, yet it is also extremely inefficient, so that improvements and optimizations in AD have evolved in the direction away of this formula.

The first computational improvement was an introduction of the concept of normalized derivatives denoted with square brackets:

$$\left[ \frac{d}{dt} \right]^n u = u^{[n]} \stackrel{\text{def}}{=} \frac{u^{(n)}}{n!}$$

The normalized  $n$ -order derivatives therefore absorb the factorial factors, becoming the Taylor coefficients, so that the rule of addition of the orders gets modified:

$$u^{[m+n]} \neq (u^{[m]})^{[n]} = \frac{(m+n)!}{m!n!} u^{[m+n]}.$$

In notation of normalized derivatives, the fundamental formulas for  $n$ -order derivatives of a product (Leibniz), quotient and power became simpler (doing away with the combinatorial factors) [Moore 1966]:

$$(1.3) \quad \begin{aligned} w &= au + bv, & w^{[n]} &= au^{[n]} + bv^{[n]} \\ w &= uv, & w^{[n]} &= \sum_{i=0}^n u^{[i]} v^{[n-i]} \end{aligned}$$

$$(1.4) \quad r = \frac{u}{v}, \quad r^{[n]} = \frac{1}{v} \left( u^{[n]} - \sum_{i=0}^{n-1} r^{[i]} v^{[n-i]} \right)$$

$$(1.5) \quad p = u^\alpha, \quad p^{[n]} = \frac{1}{u} \sum_{i=0}^{n-1} \left( \alpha \left( 1 - \frac{i}{n} \right) - \frac{i}{n} \right) p^{[i]} u^{[n-i]}; \quad n \geq 1$$

where  $a$ ,  $b$ ,  $\alpha$  are constants.

The Leibniz formula for an arbitrary number  $m$  of factors in the normalized notation simplifies too:

$$(1.6) \quad w = u_1 u_2 \dots u_m, \quad w^{[n]} = \sum_{i_1+i_2+\dots+i_m=n} u_1^{[i_1]} u_2^{[i_2]} \dots u_m^{[i_m]}.$$

Comparing the multi-factor formula (1.6) vs. its two-factor version (1.3) observe that the two factor formula requires the minimal amount of operation  $2n = O(n)$ , as well as formulas (1.4, 1.5) for a quotient and power. On the contrary, for a number of factors  $m > 2$  this amount is  $m C_n^{m-1} = O(n^{m-1})$  quickly growing with  $m$ . It represents an intermediate computational complexity between the extreme case of the Faa di Bruno and the simplest Leibniz formula.

It was not until the 50s that Steffensen [1957] probably first recognized the importance of introduction of the so called auxiliary variables and decomposition of expressions in the right hand parts of ODEs into a certain system of explicit equations containing only one (non-linear) arithmetic operation each, guaranteeing applicability of the optimal formulas (1.3-1.5). It was this idea of Steffensen that became the mantra of the Automatic Differentiation: one (non-linear) operation at a time. First used only for specific ODEs in the field of celestial mechanics, this fundamental idea of decomposition appeared applicable to virtually any ODEs

used in application due to the fact that they convert to rational right hand sides, discussed in the following chapters.

A curious reader may pose a question (like the late Prof. Michael Lidov did in the 1980s): what is so special in all those optimized formulas (1.3-1.5) vs. the general formula of Faa di Bruno? The answer is that the huge variety of possible monomials in the formula of Faa de Bruno dramatically reduces in the optimized formulas. First, because this variety of partial derivatives  $\frac{\partial^{n_1+n_2+\dots+n_m}}{\partial u_1^{n_1} \partial u_2^{n_2} \dots \partial u_m^{n_m}} f(u_1, \dots, u_m)$  in general case reduces to partials over only two variable  $u, v$ , and second, these partials are applied not to a general function  $f$ , but to a polynomial of degree 2, so that the orders higher than 2 disappear.

## CHAPTER 2

### The Definitions

Ramon Moore [1960] introduced the definition of elementary functions as functions of *one* variable, satisfying explicit systems of ODEs with rational right hand sides. As to multivariate functions, he considered rational functions (elementary by definition) and compositions formed with rational and elementary functions of one variable. As we will see, there exist also multivariate elementary functions not reducible to the rational (item 7 in Table 1).

DEFINITION 1. (*Elementariness in one variable*). A vector-function  $\mathbf{f} : \mathbf{C}^m \rightarrow \mathbf{C}^n$ ,  $\mathbf{f} = \{f_k(x_1, \dots, x_m)\}$ ,  $k = 1, \dots, n$ , is called elementary in a variable  $x_i$  near an initial point  $(a_1, \dots, a_m)$  if it is regular at this point and there exists a regular initial value problem for a system of  $N \geq n$  autonomous ordinary differential equations

$$(2.1) \quad \begin{cases} \frac{\partial u_k}{\partial x_i} = R_{ki}(u_1, \dots, u_n, \dots, u_N); & u_k|_{x_i=a_i} = f_k(a_1, \dots, a_m), \quad k = 1, \dots, N, \end{cases}$$

with rational right hand sides  $R_{ki}$  satisfied by  $\{f_k(x_1, \dots, x_m)\}$ . The components  $u_k$  are viewed as functions of  $x_i$ , all the remaining variables being considered as parameters.

DEFINITION 2. (*Elementariness in all variables*). A vector-function  $\mathbf{f} : \mathbf{C}^m \rightarrow \mathbf{C}^n$ ,  $\mathbf{f} = \{f_k(x_1, \dots, x_m)\}$ ,  $k = 1, \dots, n$ , is called elementary in all its variables  $(x_1, \dots, x_m)$  near an initial point  $(a_1, \dots, a_m)$  if it is elementary in each variable  $x_i$  ( $i = 1, \dots, m$ ) established by  $m$  systems of ordinary differential equations (2.1). A matrix  $R = \|R_{ki}\|_{k,i=1}^{N,m}$  comprised of their right hand sides is called a matrix of elementariness.

REMARK 1. Each of the IVPs (for each of the variables  $x_i$ ) generally contains its individual numbers  $N_i$  of equations. It is our convention for the sake of uniformity to fill in smaller of those systems with "dummy" components  $\frac{\partial u_k}{\partial x_i} = 0$ ,  $k = N_i + 1, \dots, N$  in order that in all of the systems the number of ODEs be equalized to  $N$ .

REMARK 2. As each of the systems (2.1) is autonomous, it "hides" the integration variable  $x_i$  as one of the functions  $u_k$  satisfying the trivial ODE  $\frac{\partial u_k}{\partial x_i} = 1$ . System (2.1) also "hides" all the parameters  $x_j \neq x_i$  as functions satisfying the trivial ODE  $\frac{\partial u_k}{\partial x_i} = 0$ . Therefore the matrix of elementariness  $R$  generally may include the unit sub-matrix  $E = \|\delta_{ki}\|_{k,i=1}^m$ .

EXAMPLE 1. A multivariate polynomial is elementary in all variables everywhere (obvious).

EXAMPLE 2. A multivariate rational function is elementary in all variables everywhere except the points where it is singular (obvious).

EXAMPLE 3. Consider a function  $f(x, y) = \cos(x)e^y$ . As a function of  $x$  it is represented by a system

$$\begin{aligned} f'_x &= -g; & f|_{x=a} &= \cos(a)e^b \\ g'_x &= f; & g|_{x=a} &= \sin(a)e^b \end{aligned}$$

(omitting  $x' = 1$ ). As a function of  $y$  it is represented by a "system"

$$\begin{aligned} f'_y &= f; & f|_{y=b} &= \cos(a)e^b \\ g'_y &= 0; & g|_{y=b} &= \text{const} \quad (\text{'dummy' component}) \end{aligned}$$

(omitting  $y' = 1$ ), so that the matrix of elementariness is

$$\begin{pmatrix} -g & f \\ f & 0 \end{pmatrix}.$$

REMARK 3. It is obvious that elementariness of a particular vector-function may be established via infinitely many systems with various number of equations (containing even uncoupled equations). Thus there must exist the minimal number of the equations required for definition of a particular vector-function, which we call "complexity" of this elementary vector function. For example, the function  $w = \tan t$  is elementary together with  $u = \cos t$ ,  $v = \sin t$  in the system

$$\begin{aligned} u' &= -v \\ v' &= u \\ w' &= \frac{1}{u^2} \end{aligned}$$

(no uncoupled equations in it). Yet this  $w$  happened to be defined also via a single ODE

$$w' = w^2 + 1.$$

Elementariness of a vector-function is defined via an IVP with respect to the variable of integration in the corresponding rational system of ODEs. Say elementariness of a vector-function  $\mathbf{x}(t)$  in  $t$  is established via a rational system

$$\mathbf{x}' = \mathbf{R}(\mathbf{x}); \quad \mathbf{x}|_{t=0} = \mathbf{a}.$$

However the *general* solution of this system  $\mathbf{X}(t, \mathbf{a})$  is also a function of the initial values  $\mathbf{a}$  (and perhaps of the parameters). It is not known, under which conditions solutions of rational ODEs are elementary with respect to the initial values or to the parameters. We do know however an example of a solution proven to be non-elementary in the parameter (item 14 in Table 1).

REMARK 4. In the frame of this Definition, the property of elementariness of a vector-function is defined locally at a point of regularity of the initial values. However by integrating the ODEs, elementariness expands into other regular points of the vector-function, but only into those that are also regular points of the corresponding system of ODEs.

REMARK 5. As the property of elementariness is defined locally at a point, some functions may possibly have points of violation of elementariness at certain points. For example, in rational functions the points where the denominator is

zero are points of violation of elementariness, though trivial, because the rational functions are undefined and singular at such points. However there exist functions whose (stand alone) elementariness (see below) violates at isolated points where the function is defined and holomorphic, for example  $\frac{\sin t}{t}$  at  $t = 0$  [7].

It is worth noting that the solution of ODEs may happen to be holomorphic even at the point, which is singular in the phase space of the particular system of ODEs, thus unreachable via integration of the system (see the Chapter on "Unremovable 'removable' singularities").

REMARK 6. *Elementariness of an  $n$ -dimensional vector-function sometimes may be established only via a system of ODEs with number of equations  $N > n$ , like elementariness in  $x$  of  $f(x, y) = \cos(x)e^y$  in the example above. We then say that a (vector-)function is elementary together with the associated components: say  $\cos t$  together with  $\sin t$ .*

Is it possible to define a "stand alone" elementariness of a component of a vector-function not referring to the associated components so that the "stand alone" elementariness be equivalent to Definition 1 in certain sense?

Observe, that many examples of elementary functions defined via a first order system, also happen to satisfy IVPs for one explicit  $n$ -order rational ODE regular at the initial point. For example, instead of the system  $u' = -v, v' = u$ , we can write down a second order ODE  $u'' = -u$  defining  $\cos t$  or  $\sin t$  as well.

COROLLARY 1. *(elementariness via one  $n$ -order ODE). A function  $u(t)$  satisfying an IVP for an explicit  $n$ -order ODE*

$$u^{(n)} = r(t, u, \dots, u^{(n-1)}), \quad u^{(i)} \Big|_{t=t_0} = a_i, \quad i = 1, \dots, n-1$$

with a rational right hand side  $r$  regular at a point  $t_0$  is elementary near  $t_0$ .

PROOF. Any explicit  $n$ -order ODE is trivially convertible to an explicit first order system of ODEs by introducing new variables  $v_k = u^{(k)}$ ,  $k = 1, \dots, n$  :

$$\begin{cases} u' = v_1 \\ v'_{k-1} = v_k, & k = 2, \dots, n-1 \\ v'_n = r(t, u, v_1, \dots, v_{n-1}) \end{cases}$$

and it is regular and rational indeed, meaning that  $u(t)$  is elementary together with the associated components  $v_k(t)$ .  $\square$

Is it possible to define elementariness via one  $n$ -order ODE rather than via a system of ODEs? The next section deals with this question.

### 1. One $n$ -order ODE vs. a system of first order ODEs

A possibility to define elementariness via one  $n$ -order ODE depends on the question, whether an equivalence between a *system of first order ODEs* vs. *one  $n$ -order ODE* takes place. More specifically, whether the equivalency between an IVP for an explicit rational *system of first order ODEs* vs. a properly chosen IVP for one explicit rational  *$n$ -order ODE* exists. The equivalence is understood in the sense that a solution (say  $u_1$ ) of an IVP

$$(2.2) \quad u_1^{(n)} = f(t, u_1, \dots, u_1^{(n-1)}), \quad u_1^{(i)} \Big|_{t=t_0} = a_1^i, \quad i = 1, \dots, n-1$$

for one  $n$ -order ODE at a regular point  $t_0$  belongs to a solution vector of some regular IVP for a first order system

$$(2.3) \quad \{u'_k = g_k(t, u_1, \dots, u_m), \quad u_k|_{t=t_0} = b_k, \quad b_1 = a_1^0, \quad k = 1, \dots, m$$

and vice versa: for each component  $u_k$  of (2.3) there exists a regular IVP for one particular  $n$ -order ODE like (2.2) satisfied by  $u_k$ .

The conversion of *one explicit rational  $n$ -order ODE* into *a first order explicit system of rational ODEs* is trivial (the Corollary), but not the opposite.

In general, if we did not ask for the rational right hand sides in ODEs, for *arbitrary holomorphic* right hand sides  $f, g_k$  this equivalency does take place indeed, albeit in a trivial tautological sense. Just consider the solution  $u_1(t)$  of system (2.3), denoting  $f(t) = u'_1(t)$ . Then the equation  $u'_1(t) = f(t)$  is the required equivalent ODE (2.2).

However with the requirement of rational right hand sides, this question of equivalency remains unanswered so far, posed as *the Conjecture* [Gofen 2008] (see more in the chapters "Transforms" and "Open questions").

The next two chapters cover the main theorems and the fundamental properties of elementary functions, summarized in Table 1.



## CHAPTER 3

### The main theorems

In this Chapter we are going to establish three fundamental theorems about closedness of the class of elementary functions in the following sense. We are to prove that the composition of elementary vector-functions is elementary (Theorem 2), and the inverse to an invertible elementary vector-function is also elementary (Theorem 3).

Moreover, if the right hand sides of an explicit system of ODEs comprise an *elementary* vector-function, the solution is an elementary vector-function too. The solution therefore does belong to the same class as the right hand sides themselves. This fundamental fact is established by the following

**THEOREM 1.** (*Elementariness of solutions of ODEs*). *Let the initial value problem*

$$\{u'_k = f_k(u_1, \dots, u_m), \quad u_k|_{t=t_0} = a_k, \quad k = 1, \dots, m$$

*be regular near  $t = t_0$ , and the vector-function  $\{f_k(u_1, \dots, u_m)\}$  of the right hand sides elementary near  $(a_1, \dots, a_m)$  in all the variables. Then its solution  $\{u_1(t), \dots, u_m(t)\}$  is elementary at  $t = t_0$ , i.e. it satisfies a regular IVP for a (larger) system of rational ODEs (to be constructed in the process of the proof).*

**PROOF.** If the vector-function  $\{f_k(u_1, \dots, u_m)\}$  is already rational, the theorem is obviously true. Otherwise the fact that the vector-function  $\{f_k(u_1, \dots, u_m)\}$  is elementary near  $(a_1, \dots, a_m)$  in all the variables means that for each of the variables  $u_j$  ( $j = 1, \dots, m$ ) we can produce a particular rational system of autonomous ODEs defining  $\{f_k\}$  as a function of this variable  $u_j$  (the others being considered as parameters):

$$\left\{ \frac{\partial f_i}{\partial u_j} = R_{ij}(f_1, \dots, f_m, \dots, f_n), \quad i = 1, \dots, m, \dots, n. \right.$$

Introduce new variables

$$v_i = f_i(u_1, \dots, u_m) \quad v_i|_{t=t_0} = b_i = f_i(a_1, \dots, a_m), \quad i = 1, \dots, n.$$

Differentiate the new variables:

$$\begin{aligned} v'_i &= [f_i(u_1, \dots, u_m)]' = \sum_{j=1}^m \frac{\partial f_i}{\partial u_j} u'_j = \sum_{j=1}^m R_{ij}(f_1, \dots, f_m, \dots, f_n) v_j = \\ &= \sum_{j=1}^m R_{ij}(v_1, \dots, v_m, \dots, v_n) v_j, \quad i = 1, \dots, n. \end{aligned}$$

Merging these equations with the source ODEs, obtain the regular IVP for a closed system

$$\begin{cases} u'_k = v_k, & u_k|_{t=t_0} = a_k, & k = 1, \dots, m \\ v'_i = \sum_{j=1}^m R_{ij}(v_1, \dots, v_m, \dots, v_n)v_j, & v_i|_{t=t_0} = b_i, & i = 1, \dots, n \end{cases}$$

which proves that solution  $(u_1, \dots, u_m)$  is elementary as a part of a larger vector-function  $(u_1, \dots, u_m, v_1, \dots, v_n)$ .  $\square$

REMARK 7. *It is always possible therefore to convert a non-rational (transcendental) system of elementary ODEs to a larger rational system of ODEs (say, to rid of transcendental right hand sides). The number of the added equations does not exceed the number  $n$  of rows in the matrix of elementariness for the (non-rational) right hand sides of the source system. In particular cases however the total number of equations may happen to be less than  $m + n$  due to simplifications.*

EXAMPLE 4. *Following the reasoning of the Theorem, the elementary (transcendental) IVP  $u' = e^u$ ,  $u|_{t=0} = a$  converts to the rational system*

$$\begin{aligned} u' &= v, & u|_{t=0} &= a \\ v' &= v^2, & v|_{t=0} &= e^a. \end{aligned}$$

*Simplification is possible here: actually  $v = \frac{1}{e^{-a}-t}$  so that  $u$  is defined via one rational ODE*

$$u' = \frac{1}{e^{-a}-t}, \quad u|_{t=0} = a$$

*having the solution  $u = -\ln(e^{-a} - t)$ .*

The fact that a composition of elementary vector-functions is elementary vector-function is established in the following

THEOREM 2. *(Elementariness of a composition of functions). Let the vector-function  $\mathbf{y} : \mathbf{C}^k \rightarrow \mathbf{C}^m$ ,  $\mathbf{y} = \{y_j(x_1, \dots, x_k)\}$ ,  $j = 1, \dots, m$ , be elementary near  $(x_1, \dots, x_k)$  in all its variables, and a function  $f_1(y_1, \dots, y_m) : \mathbf{C}^m \rightarrow \mathbf{C}^1$  be elementary near  $(y_1, \dots, y_m)$  in all its variables too. Then the composition*

$$f_1(y_1(x_1, \dots, x_k), \dots, y_m(x_1, \dots, x_k)) = z_1(x_1, \dots, x_k)$$

*is elementary near  $(x_1, \dots, x_k)$  in all its variables.*

PROOF. Let elementariness of the function  $f_1$  be established by a matrix  $P = \|p_{ij}\|_{i=1, j=1}^{i=n, j=m}$  of rational functions  $p_{ij}(f_1, \dots, f_n)$ , and elementariness of the vector-function  $\mathbf{y}$  - by a matrix  $Q = \|q_{jl}\|_{j=1, l=1}^{j=M, l=k}$ ,  $M \geq m$  of rational functions  $q_{jl}(y_1, \dots, y_M)$ . Denote the compositions

$$f_i(y_1(x_1, \dots, x_k), \dots, y_m(x_1, \dots, x_k)) = z_i(x_1, \dots, x_k), \quad i = 1, \dots, n.$$

Then for each of the variables  $x_l$ ,  $l = 1, \dots, k$

$$\begin{aligned} \frac{\partial z_i}{\partial x_l} &= \sum_{j=1}^m \frac{\partial f_i}{\partial y_j} \frac{\partial y_j}{\partial x_l} = \sum_{j=1}^m p_{ij}(f_1, \dots, f_n) q_{jl}(y_1, \dots, y_M) = \\ &= \sum_{j=1}^m p_{ij}(z_1, \dots, z_n) q_{jl}(y_1, \dots, y_M), \quad i = 1, \dots, n \end{aligned}$$

so that for each  $x_l$ ,  $l = 1, \dots, k$  we obtained a closed system whose right hand sides are all rational

$$\begin{cases} \frac{\partial z_i}{\partial x_l} = \sum_{j=1}^m p_{ij}(z_1, \dots, z_n) q_{jl}(y_1, \dots, y_M), & i = 1, \dots, n \\ \frac{\partial y_j}{\partial x_l} = q_{jl}(y_1, \dots, y_M), & j = 1, \dots, M \end{cases}$$

thus establishing elementariness of  $z_1(x_1, \dots, x_k)$  in all its variables. These right hand sides for all variables  $x_l$ ,  $l = 1, \dots, k$  may be organized into a matrix

$$R = \begin{vmatrix} PQ_0 \\ Q \end{vmatrix}, \quad Q_0 = \|q_{jl}\|_{j=1}^{j=m, l=k}$$

( $Q_0$  is an  $m \times k$  sub-matrix of  $M \times k$  matrix  $Q$ ).  $\square$

REMARK 8. *The number  $M + n$  of equations establishing elementariness of the composition equals to the sum of those numbers for the vector-functions  $\mathbf{y}$  and  $\mathbf{z}$ .*

REMARK 9. *There are examples of functions looking as though a superposition of elementary functions but whose elementariness violates at certain point. For example, both functions  $\sin t$  and  $t$  are elementary everywhere, yet as it was proven, the stand alone elementariness of  $\frac{\sin t}{t}$  violates at  $t = 0$ . However the functions  $\frac{\sin t}{t}$  is a composition of a rational function  $R(x, y) = \frac{x}{y}$  and elementary functions  $\sin t$  and  $t$ . The rational function  $R$  is not elementary at points where  $y = 0$ , therefore no contradiction to the Theorem takes place.*

Elementariness of the inverse vector functions is established by the following

THEOREM 3. *(Elementariness of an inverse vector-function). Let the vector-function  $\mathbf{y} : \mathbf{C}^m \rightarrow \mathbf{C}^m$ ,  $\mathbf{y} = \{y_k(x_1, \dots, x_m)\}$ ,  $k = 1, \dots, m$ , be invertible and elementary near  $(x_1, \dots, x_m)$  in all its variables. Then the inverse vector-function  $\mathbf{X}$  is elementary in the respective neighborhood of  $(y_1, \dots, y_m)$  in all the variables too.*

PROOF. Let  $\mathbf{X} = \{X_i(y_1, \dots, y_m)\}$  be the inverse to  $\mathbf{y}$ , and let elementariness of  $\mathbf{y}$  be established by the matrix  $R = \|r_{ki}\|_{k,i=1}^{k=M, i=m}$ ,  $M \geq m$  of rational functions  $r_{ki}(y_1, \dots, y_m, \dots, y_M)$ . That means that for each  $x_i$

$$\begin{cases} \frac{\partial y_k}{\partial x_i} = r_{ki}(y_1, \dots, y_m, \dots, y_M), & k = 1, \dots, m, \dots, M. \end{cases}$$

Denote  $R_0 = \|r_{ki}\|_{k,i=1}^{k=m, i=m}$  (a square  $m \times m$  sub-matrix of  $M \times m$  matrix  $R$ ), and  $R_1 = \|r_{ki}\|_{k=m+1, i=1}^{k=M, i=m}$  so that

$$R = \begin{vmatrix} R_0 \\ R_1 \end{vmatrix}.$$

Observe that  $R_0$  represents a Jacobian of the vector function  $\mathbf{y}$ , thus it must be regular at point  $(x_1, \dots, x_m)$  because the vector-function  $\mathbf{y}$  is invertible, so that  $R_0^{-1}$  exists. Therefore

$$\left\| \frac{\partial X_i}{\partial y_k} \right\| = \left\| \frac{\partial y_k}{\partial x_i} \right\|^{-1} = R_0^{-1}.$$

Obtained by inversion of rational matrix  $R_0$ , the elements of  $R_0^{-1} = \|q_{ik}\|_{i,k=1}^{i=m, k=m}$  must all be rational functions too. So for each  $y_k$  we have a system of ODEs

$$\begin{cases} \frac{\partial X_i}{\partial y_k} = q_{ik}(y_1, \dots, y_m, \dots, y_M), & i = 1, \dots, m \end{cases}$$

in unknown functions  $X_i$  and  $y_k$ : not yet closed. Considering  $y_k$  an independent variable, obtain ODEs for all remaining  $y_j$  :

$$\frac{\partial y_j}{\partial y_k} = \sum_{l=1}^m \frac{\partial y_j}{\partial x_l} \frac{\partial x_l}{\partial y_k} = \sum_{l=1}^m \frac{\partial y_j}{\partial x_l} \frac{\partial X_l}{\partial y_k} = \sum_{l=1}^m r_{jl} q_{lk}, \quad j = 1, \dots, m, \dots, M.$$

The right hand sides of these systems therefore may be presented as a matrix

$$RR_0^{-1} = \begin{vmatrix} R_0 \\ R_1 \end{vmatrix} R_0^{-1} = \begin{vmatrix} E \\ R_1 R_0^{-1} \end{vmatrix}$$

so that only the equations for  $j = m + 1, \dots, M$  are non-trivial. Finally, for each  $y_k$  the closed system

$$(3.1) \quad \begin{cases} \frac{\partial y_j}{\partial y_k} = \delta_{jk}, & j = 1, \dots, m, \\ \frac{\partial y_j}{\partial y_k} = \sum_{l=1}^m r_{jl}(y_1, \dots, y_M) q_{lk}(y_1, \dots, y_M), & j = m + 1, \dots, M \\ \frac{\partial X_i}{\partial y_k} = q_{ik}(y_1, \dots, y_M), & i = 1, \dots, m \end{cases}$$

establishes elementariness of the vector function  $\mathbf{X}$  in all its variables  $y_1, \dots, y_m$ . The matrix of elementariness is

$$Q = \begin{vmatrix} RR_0^{-1} \\ R_0^{-1} \end{vmatrix} = \begin{vmatrix} E \\ R_1 R_0^{-1} \\ R_0^{-1} \end{vmatrix}$$

□

REMARK 10. Observe that ODEs (3.1) for components  $X_i$  of the inverse vector-functions are not coupled. Thus we can write down smaller closed systems of ODEs for each of the components  $X_i$  separately ( $i$  does not run from 1 through  $m$ , but takes only one value).

COROLLARY 2. For  $m = 1$ , when  $y_1(t)$  is elementary satisfying the system

$$\{y_i' = r_i(y_1, \dots, y_M), \quad i = 1, \dots, M,$$

the inverse function  $T(y_1)$  is elementary satisfying the system

$$\begin{cases} \frac{dT}{dy_1} = \frac{1}{r_1(y_1, \dots, y_M)} \\ \frac{dy_i}{dy_1} = \frac{r_i(y_1, \dots, y_M)}{r_1(y_1, \dots, y_M)}, & i = 2, \dots, M. \end{cases}$$

COROLLARY 3. Let the vector-function  $\mathbf{y} : \mathbf{C}^m \rightarrow \mathbf{C}^m$ ,  $\mathbf{y} = \{y_j(x_1, \dots, x_m; p)\}$ ,  $j = 1, \dots, m$ , with a parameter  $p$  be invertible and elementary near  $(x_1, \dots, x_m; p)$  in all its variables including the parameter. Then the inverse vector-function  $\mathbf{X}$  is elementary in the respective neighborhood of  $(y_1, \dots, y_m; p)$  in all its variables including the parameter too.

PROOF. The method of the proof and the ODEs establishing elementariness in  $(y_1, \dots, y_m)$  remain the same as in the Theorem, because the parameter does not occur in the ODEs explicitly. The fact that the source vector  $\mathbf{y}$  is elementary in  $p$  is given by the system of corresponding rational ODEs

$$\left\{ \frac{\partial y_k}{\partial p} = r_k(y_1, \dots, y_m, \dots, y_M), \quad k = 1, \dots, m, \dots, M. \right.$$

Now consider the inverse vector  $\mathbf{X}(y_1, \dots, y_m)$  depending on  $p$  only via  $(y_1, \dots, y_m)$ :

$$\begin{aligned} \frac{\partial X_i}{\partial p} &= \sum_{l=1}^m \frac{\partial X_i}{\partial y_l} \frac{\partial y_l}{\partial p} = \\ &= \sum_{l=1}^m q_{il}(y_1, \dots, y_m, \dots, y_M) r_l(y_1, \dots, y_m, \dots, y_M). \end{aligned}$$

The closed system establishing elementariness of the inverse vector in parameter  $p$  is:

$$\left\{ \begin{array}{l} \frac{\partial y_k}{\partial p} = r_k(y_1, \dots, y_m, \dots, y_M), \quad k = 1, \dots, m, \dots, M. \\ \frac{\partial X_i}{\partial p} = \sum_{l=1}^m q_{il}(y_1, \dots, y_m, \dots, y_M) r_l(y_1, \dots, y_m, \dots, y_M) \end{array} \right.$$

□

REMARK 11. *While elementariness in the parameters of the general solution of elementary ODEs is an open question, for the inverse vector-function elementariness in the parameter is proved.*



## CHAPTER 4

### Properties of elementary functions

The main properties of elementary functions are summarized in Table 1 below. Items 3-5, 9 were proved in the chapter Main Theorems.

Item 6: Multivariate algebraic functions are elementary as inverse to the polynomial functions.

Item 7: Let elementariness of  $u(t)$  be established by the system  $\mathbf{u}' = \mathbf{R}(\mathbf{u}, t)$ . Then the derivative  $\mathbf{u}' = \mathbf{v}$  satisfies a rational equation  $\mathbf{v}' = \sum \frac{\partial \mathbf{R}}{\partial u_i} R_i + \frac{\partial \mathbf{R}}{\partial t}$ .

Item 8: The antiderivative  $u(t, x) = \int f(t, x) dt$  satisfies the ODE  $u'_t = f(t, x)$  where  $x$  is a parameter and  $f$  is elementary. Therefore  $u$  is elementary in  $t$  by Theorem 1, but it can happen to be non-elementary in  $x$  like the Gamma integral (Item 14).

Item 10 : Consider the Gamma function  $\Gamma(t)$  at a regular point  $t_0$  and its Taylor expansion. The partial sums  $S_n(t)$  of the Taylor expansion are elementary as polynomials and they do converge:  $\lim_{n \rightarrow \infty} S_n(t) = \Gamma(t)$ . Yet  $\Gamma(t)$  is later proven non-elementary.

Items 11-15 are discussed farther in the article.

1	Polynomial functions	Elementary in all variables everywhere being trivial examples of multivariate elementary functions (see item 7 for the non-trivial)
2	Rational functions	Elementary in all variables everywhere except its points of singularity, being trivial examples of multivariate elementary functions (see item 7 for the non-trivial)
3	Conventional elementary functions and some special such as Airy's	Elementary
4	Composition of elementary in all variables vector-functions	Elementary in all variables (Theorem 2)
5	Inverse to an elementary in all variables vector-function	Elementary in all variables (Theorem 3)
6	An inverse function $X_n(x_1, \dots, x_{n-1})$ defined via an implicit equation $F(x_1, \dots, x_n) = 0$ , where $F$ is elementary in all variables	Elementary in all variables (Corollary 3)
7	Multivariate algebraic functions	Elementary at regular points in all variables exemplifying multivariate elementary functions generally not expressible via rational functions
8	Derivatives of an elementary function	Elementary
9	Integral $\int f(t, x)dt$ , $f$ elementary in $t, x$	Elementary in $t$ , but not necessarily in $x$ . Only one such example (item 15) proven non-elementary.
10	Vector-function $u_k(t, x)$ , a solution of an IVP $\{u'_k = f_k(u_1, \dots, u_n, x)\}$ , $\{f_k\}$ elementary in all variables	Elementary in $t$ (Theorem 1), not necessarily in $x$ . Analytical continuation in $t$ is doable into the domain of regularity of the ODEs
11	$\lim_{n \rightarrow \infty} f_n(t)$ , all $f_n(t)$ elementary	Not necessarily elementary
12	$\sum_{n=0}^{\infty} a_n t^n$ converging to $f(t)$ , $a_n$ are arbitrary	Not necessarily elementary. Generally a method of analytical continuation not known.
13	$\sum_{n=0}^{\infty} a_n t^n$ converging to $f(t)$ , $a_n$ obtained via AD formulas for elementary ODEs	Elementary. Analytical continuation is doable via integration into the domain of regularity of the ODEs
14	Euler's Gamma function $\Gamma(x)$	Not elementary everywhere
15	$G(t, x) = \int_0^t u^{x-1} e^{-u} du$ generating $\Gamma(x)$ so that $\Gamma(x) = G(\infty, x)$	Elementary in $t$ , non-elementary in $x$ for all $t > 0$



Table 1. Properties of elementary functions



## Fundamental transforms

### 1. Elementary systems - into special larger systems

In this section we are to show that all explicit first order systems of ODEs with elementary right hand sides are convertible to larger systems in special formats summarized in the Table 2 below.

(1) An explicit first order system of ODEs whose right-hand side is <b>elementary</b> vector-function converts to...	
(2) A system of ODEs whose right-hand sides are <b>rational functions</b> (Theorem 1). At regular points it further converts to...	
(3) A <b>canonical</b> system: an explicit system of algebraic and differential equations for computing $n$ -order derivatives requiring $O(n^2)$ operations.	(4) A system, whose right-hand sides are <b>polynomials</b> . It further converts to...
	(5) Polynomial ODEs of <b>degree <math>\leq 2</math></b> . It further converts to polynomial ODEs of degree 2 with ...
	(6) ...with coefficients <b>0, 1 only</b> (Kerner [6])

Table 2. Transformations of elementary systems of ODEs

We speak about conversion of the source system into a larger target system of ODEs in the sense of the equivalent transformation. That means the solution vector of the target system contains the solution vector of the source one. Some components of the target system are introduced as known relations over the source components. Therefore the initial values of the target systems are not free. The newly introduced components are functions of the source initial values, while the corresponding relations are in fact the integrals of the target system.

Let a system of ODEs

$$(5.1) \quad \{u'_k = f_k(u_1, \dots, u_m), \quad u_k|_{t=t_0} = a_k, \quad k = 1, \dots, m$$

be an elementary meaning that the vector-function  $\{f_k(u_1, \dots, u_m)\}$  is elementary in all its variables. The possibility of transformation to a rational system was proved in Theorem 1. To not complicate our notation, assume system (5.1) to be already rational.

**1.1. Canonical system.** (Table 2 cell 3).

Assuming the right hand side all rational, we can introduce the so called auxiliary variables  $u_{m+1}, u_{m+2}, \dots$  in a sequence of explicit equations with a goal that only one arithmetic operation is allowed in the right hand sides:

$$(5.2) \quad \begin{cases} u_k = r_k(u_\alpha, u_\beta), & \alpha, \beta < k, & k = m+1, \dots, M \\ u'_k = u_\alpha, & \alpha \leq M, & k = 1, \dots, m. \end{cases}$$

Here  $r_k(u_\alpha, u_\beta)$  stands for a rational function implementing one of the four arithmetic operations over no more than two operands, so that the optimized formulas (1.3-1.5) of  $n$ -order differentiation can be applied. This format is neither unique, nor the shortest possible<sup>1</sup>.

The canonical form is the main instrument of Automatic Differentiation for iterative computing of  $n$ -order derivatives. At  $i$ -th iteration, formulas (1.3-1.5) require  $O(i)$  operations,  $i = 1, \dots, n$ , which sums up to  $O(\sum_{i=1}^n i) = O(n^2)$ . It is due to this estimate that AD and the modern Taylor method are practically feasible and compete with other methods of numeric integration.

The rest of transformations (4-7) are important in theoretical considerations.

**1.2. Conversion to polynomial systems.** (Table 2 cell 4).

Assume system (5.1) to be already rational and regular at an initial point so that

$$\begin{cases} u'_k = \frac{p_k(u_1, \dots, u_m)}{q_k(u_1, \dots, u_m)}, & u_k|_{t=t_0} = a_k, & k = 1, \dots, m. \end{cases}$$

Introduce new variables

$$v_k = \frac{1}{q_k(u_1, \dots, u_m)}.$$

Then

$$\begin{aligned} u'_k &= p_k(u_1, \dots, u_m)v_k, & u_k|_{t=t_0} &= a_k, & k &= 1, \dots, m. \\ v'_k &= -v_k^2 \left( \frac{\partial q_k}{\partial u_1} u'_1 + \dots + \frac{\partial q_k}{\partial u_m} u'_m \right) = \\ &= -v_k^2 \left( \frac{\partial q_k}{\partial u_1} p_1 v_1 + \dots + \frac{\partial q_k}{\partial u_m} p_m v_m \right), & v_k|_{t=t_0} &= \frac{1}{q_k(a_1, \dots, a_m)} \end{aligned}$$

is a polynomial system. It is not necessarily the optimal<sup>1</sup>.

The further transforms (cells 5-7 in the Table) to polynomials of degree 2 and the special formats were first introduced in the papers [Charnyi 1970] and [Kerner 1981] independently. Both authors perhaps were not aware of the Moore's concept of elementary functions and about the *uniform* method of converting elementary ODEs into the rational ones (as in Theorem 1). They did mention however that the conventional elementary functions satisfy rational ODEs, and ODEs in applications are comprised of the conventional elementary functions.

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<sup>1</sup>A shorter system may be obtained by identifying the common sub-expressions.

**1.3. Conversion to polynomial systems of second degree.** (Table 2 cell 5).

Assume that the source system (5.1) is already converted to the polynomial system

$$(5.3) \quad u'_k = P_k(u_1, \dots, u_m) = \sum_{(\alpha, \beta, \dots, \omega)} b_{\alpha\beta\dots\omega}^k u_1^\alpha \dots u_m^\omega, \quad u_k|_{t=t_0} = a_k, \quad k = 1, \dots, m$$

and the maximal degree of the polynomials  $P_k$  is  $r$ . Consider a set of all monomials  $u_1^\alpha \dots u_m^\omega$  of a degree  $\leq r$ : the number of such monomials is  $m^r$ . Introduce  $m^r$  new variables  $v_{\alpha\beta\dots\omega} = u_1^\alpha \dots u_m^\omega$ . In these new variables equations (5.3) become linear:

$$(5.4) \quad u'_k = \sum b_{\alpha\beta\dots\omega}^k v_{\alpha\beta\dots\omega}, \quad k = 1, \dots, m.$$

Obtain derivatives of the newly introduced variables:

$$(5.5) \quad \begin{aligned} v'_{\alpha\beta\dots\omega} &= (u_1^\alpha \dots u_m^\omega)' = \\ &= \alpha(u_1^{\alpha-1} \dots u_m^\omega)u'_1 + \dots + \omega(u_1^\alpha \dots u_m^{\omega-1})u'_m = \\ &= \alpha v_{\alpha-1, \beta \dots \omega} u'_1 + \dots + \omega v_{\alpha \beta \dots \omega-1} u'_m = \\ &= \alpha v_{\alpha-1, \beta \dots \omega} \sum b_{\alpha\beta\dots\omega}^1 v_{\alpha\beta\dots\omega} + \dots + \omega v_{\alpha\beta\dots\omega-1} \sum b_{\alpha\beta\dots\omega}^m v_{\alpha\beta\dots\omega}. \end{aligned}$$

Equations (5.4, 5.5) comprise a closed polynomial system of second degree. Denote multi-index variables  $v_{\alpha\beta\dots\omega}$  in linearly indexed variables  $w_k$ ,  $k$  running from 1 to  $N = m^r$ . In this new notation, rewrite system (5.4, 5.5) as a general second degree polynomial system

$$(5.6) \quad \left\{ w'_k = \sum_{i, j} b_{ij}^k w_i w_j + \sum_i b_i^k w_i + c^k, \quad k = 1, \dots, N. \right.$$

This second degree polynomial system may be converted to a second degree form by introducing a dummy variable  $w_k : w'_k = 0, \quad w_k|_{t=t_0} = 1, \quad (k = N+1)$  and using it as a factor in the equations. To not complicate the notation, rewrite equation (5.6) as a form with the same coefficients

$$(5.7) \quad \left\{ w'_k = \sum_{i, j=1}^N b_{ij}^k w_i w_j, \quad k = 1, \dots, N. \right.$$

**1.4. Conversion to a system with degree 2 forms in zeros and ones only.** (Table 2, cell 6, [Kerner 1981]).

Introduce  $N^4$  new variables  $x_{ijkl}$  in system (5.7) through relations

$$x_{ijkl} = b_{ij}^k w_l, \quad i, j, k, l = 1, \dots, N.$$

Differentiate them taking into consideration equations (5.7):

$$\begin{aligned} x'_{ijkl} &= b_{ij}^k w'_l = b_{ij}^k \sum_{\alpha, \beta=1}^N b_{\alpha\beta}^l w_\alpha w_\beta = \sum_{\alpha, \beta=1}^N b_{ij}^k w_\alpha b_{\alpha\beta}^l w_\beta \\ &= \sum_{\alpha, \beta=1}^N x_{ijk\alpha} x_{\alpha\beta l}, \quad i, j, k, l = 1, \dots, N. \end{aligned}$$

This is a closed system with degree 2 forms, each containing zeros and ones only (not all possible monomials  $x_{ijk\alpha}x_{\alpha\beta l\beta}$  occur in each of the forms).

**1.5. Conversion to a system in squares only.** (Table 2, cell 7 [Charnyi 1970]).

Introduce  $C_N^2$  new variables  $x_{ij}, y_{ij}$  for each pair of different  $w_i, w_j$  in system (5.7) through relations

$$\begin{aligned} w_i &= x_{ij} + y_{ij}, & x_{ij} &= (w_i + w_j)/2 \\ w_j &= x_{ij} - y_{ij}, & y_{ij} &= (w_i - w_j)/2 \end{aligned}$$

so that system (5.7) now contains squares only,

$$w'_k = \sum_{i,j=1}^N b_{ij}^k (x_{ij}^2 - y_{ij}^2), \quad k = 1, \dots, N.$$

To close the system, obtain ODEs for  $x_{ij}, y_{ij}$  :

$$\begin{aligned} x'_{ij} &= (w_i + w_j)/2 = \frac{1}{2} \sum_{\alpha, \beta=1}^N (b_{\alpha\beta}^i + b_{\alpha\beta}^j)(x_{\alpha\beta}^2 - y_{\alpha\beta}^2) \\ y'_{ij} &= (w_i - w_j)/2 = \frac{1}{2} \sum_{\alpha, \beta=1}^N (b_{\alpha\beta}^i - b_{\alpha\beta}^j)(x_{\alpha\beta}^2 - y_{\alpha\beta}^2) \end{aligned}$$

Rewrite these three groups of equations in  $w_i, x_{ij}, y_{ij}$  into linearly indexed variables  $z_k, k = 1, \dots, N, \dots, M$  :

$$\left\{ z'_k = \sum_{i=1}^M c_i^k z_i^2, \quad k = 1, \dots, M \right.$$

which is a polynomial system in squares only.

The methods of transforms (4-7) performed above are neither unique, nor necessarily optimal. They were chosen for the reason of uniformity and simplicity only.

## 2. Elementary systems - into one $n$ -order ODE

In the previous section we considered transforms of the first order system into larger systems with special features. Is an "opposite" transform, from a system to one ODE (of an order  $n$ ) possible too?

In the chapter "Definitions" we posed a question whether the equivalency between a regular IVP for a system of first order ODEs vs. a regular IVP for one ODE of order  $n$  really takes place in the class of ODEs with rational right hand parts. As of today, the complete solution of this problem is not known. A partial solution follows.

**THEOREM 4.** *For each component  $u_k$  of the solution of a regular IVP (5.1) with elementary right hand sides there exists an  $n$ -order implicit polynomial ODE*

$$P(t, u, u', \dots, u^{(n)}) = 0$$

*with a nonzero polynomial*

$$P(T, U_0, U_1, \dots, U_n)$$

*satisfied by  $u = u_k(t)$ .*

REMARK 12. *The Theorem does not claim however as though*

$$\left. \frac{\partial P(t, U_0, U_1, \dots, U_n)}{\partial U_n} \right|_{t=t_0} \neq 0,$$

*which would be equivalent to the Conjecture [Gofen 2008] not yet proved.*

PROOF. Assume that the source system (5.1) is already polynomial. For simplicity consider it in variables  $u, v, w$  and polynomials  $F, G, H$

$$(5.8) \quad \begin{aligned} u' &= F(t, u, v, w) \\ v' &= G(t, u, v, w) \\ w' &= H(t, u, v, w) \end{aligned}$$

with the goal of obtaining an  $n$ -order implicit polynomial ODE satisfied by  $u$ . Repeatedly differentiate the equation for  $u$  obtaining an infinite sequence

$$(5.9) \quad \begin{aligned} u' &= F_1(t, u, v, w) = F(t, u, v, w) \\ u'' &= F_2(t, u, v, w) = \frac{\partial F_1}{\partial T} + \frac{\partial F_1}{\partial U} F + \frac{\partial F_1}{\partial V} G + \frac{\partial F_1}{\partial W} H \\ &\dots \\ u^{(n)} &= F_n(t, u, v, w) = \frac{\partial F_{n-1}}{\partial T} + \frac{\partial F_{n-1}}{\partial U} F + \frac{\partial F_{n-1}}{\partial V} G + \frac{\partial F_{n-1}}{\partial W} H \\ &\dots \end{aligned}$$

of nonzero<sup>2</sup> polynomials  $F_n(t, u, v, w)$ ,  $n = 1, 2, \dots$ . Consider equations (5.9) as an algebraic system over  $(t, u, v, w)$  treating  $u^{(n)}$  as parameters

$$F_n(T, U, V, W) - U_n = A_n(T, U, V, W, U_n) = 0, \quad n = 1, 2, \dots$$

This system does have a solution (the solution of the IVP). In order to rid of  $V, W$ , apply algebraic elimination. To rid of  $W$ , form the resultants  $\langle A_1, A_2 \rangle = B_1$ ,  $\langle A_1, A_3 \rangle = B_2$ , obtaining a new system without  $W$ :

$$\begin{aligned} 0 &= B_1(T, U, V, U_1, U_2) \\ 0 &= B_2(T, U, V, U_1, U_3) \end{aligned}$$

To rid of  $V$ , form the resultants  $\langle B_1, B_2 \rangle = P(T, U, U_1, U_2, U_3)$ , with polynomial  $P$  delivering the required ODE.  $\square$

Disregarding impracticality of the method of resultants, this Theorem establishes the possibility of transformation of an explicit polynomial first order ODEs into one implicit polynomial ODE of order  $n$ , leaving open the question about possible singularity of this ODE at the initial point.

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<sup>2</sup>If one - and therefore all subsequent  $F_n$  were zero polynomials, the solution  $u$  ought to be a polynomial in  $t$ , surely a solution of a polynomial ODE.





## Non-elementariness of Gamma function and Gamma integral

**THEOREM 5.** *At any point  $x$ , the Euler's Gamma function  $\Gamma(x)$  is not elementary.*

**PROOF.** Suppose it is, i.e. it satisfies some rational system (5.1). As it was showed in the previous chapter, rational system (5.1) converts to an implicit polynomial ODE

$$P(t, u, u', \dots, u^{(n)}) = 0$$

so that  $\Gamma(x)$  must satisfy this ODE, which contradicts to the theorem of Gölder [Gelfond, 1971]. According to it,  $\Gamma(x)$  can not be a solution of any polynomial ODE of any order, which proves the Theorem.  $\square$

Now consider the Gamma integral

$$G(t, x) = \int_0^t \tau^{x-1} e^{-\tau} d\tau.$$

**THEOREM 6.** *At any point  $x$ , the function  $G(t, x)$  is not elementary in  $x$  for any  $t > 0$ .*

**PROOF.** Suppose it is, i.e.  $G(t, x) = G_1(t, x)$  satisfies a rational system near some point  $(t, x)$ ,  $t > 0$

$$\left\{ \frac{\partial G_k(t, x)}{\partial x} = R_k(G_1, \dots, G_m) \right\}.$$

It converts to implicit polynomial ODE

$$P\left(x, G(t, x), \frac{\partial G(t, x)}{\partial x}, \dots, \frac{\partial^n G(t, x)}{\partial x^n}\right) = 0.$$

Consider the limits

$$\lim_{t \rightarrow \infty} \frac{\partial^n G(t, x)}{\partial x^n} = \frac{d^n \Gamma(x)}{dx^n}$$

existing for any  $x$  except the poles of the Gamma functions. Now apply the  $\lim_{t \rightarrow \infty}$  : first to the polynomial  $P$ , and then - directly to its variables (which is possible

because polynomial function is continuous):

$$\begin{aligned} & \lim_{t \rightarrow \infty} P \left( x, G(t, x), \frac{\partial G(t, x)}{\partial x}, \dots, \frac{\partial^n G(t, x)}{\partial x^n} \right) = \\ & = \left( x, \lim_{t \rightarrow \infty} G(t, x), \lim_{t \rightarrow \infty} \frac{\partial G(t, x)}{\partial x}, \dots, \lim_{t \rightarrow \infty} \frac{\partial^n G(t, x)}{\partial x^n} \right) = \\ & = P \left( x, \Gamma(x), \frac{\partial \Gamma(x)}{\partial x}, \dots, \frac{\partial^n \Gamma(x)}{\partial x^n} \right) \end{aligned}$$

which is impossible due to the previous Theorem.  $\square$

These two are so far the only examples of non-elementary functions: non-elementary at all points of their regularity. Other solutions of non-linear finite difference equations, or some solutions of ODEs as functions on parameters, may be suspects for being non-elementary also - until something similar to the Gölder Theorem for  $\Gamma(x)$  is established for them too.

Functions which are suspects for having *isolated* non-elementary points are discussed in the next chapter.

## Regular solutions of singular ODEs

Analytical continuation of elementary functions usually happens via integration of rational systems of ODEs *regular* in the points of the phase space, so that the Taylor coefficients are obtainable by evaluation of these explicit ODEs or the canonical equations. This way, analytical continuation of the solution is possible into all points where the solution is holomorphic *and* the ODEs are regular. However the latter is not always the case.

The solutions may be holomorphic at a particular point, which happened to be singular for the corresponding ODEs. For example, such is the point  $t = 0$  for the solution  $x = t^n$  ( $n$  - natural) of the IVP

$$x' = \frac{nx}{t}; \quad x|_{t=1} = 1,$$

not reachable therefore via integration of this ODE. Indeed, for a solution as simple as this, we can produce a trivial regular ODE  $x' = nt^{n-1}$  integrable into any point.

However there exist holomorphic (actually entire) functions with such a particular point, at which *no* rational ODE satisfied by this function can be regular [Gofen 2008, Flanders 2007]. For example, such are the solutions of the following singular ODEs:

$$tx'' - x = 0; \quad x|_{t=0} = 0; \quad x'|_{t=0} = 1,$$

or

$$tx' - tx + x - 1 = 0; \quad x|_{t=0} = 1; \quad \left( x(t) = \frac{e^t - 1}{t} \text{ is the solution} \right).$$

If we knew that these functions cannot satisfy also any explicit *system* of rational ODEs regular at  $t = 0$ , it would mean they are examples of holomorphic functions non-elementary at one isolated point. Meanwhile they are only suspects for being non-elementary at the point  $t = 0$ .

REMARK 13. *Unlike  $\Gamma(z)$  which can not satisfy any  $n$ -order polynomial ODE, these two functions do satisfy polynomial ODEs. However these ODEs happen to have singularity at the point, where the solution is regular.*

### 1. When explicit evaluation of the Taylor coefficients fails

In explicit rational ODEs the only possible source of singularities in the right hand sides are zero denominators (not to be confused with possible singularities of the solution itself). Consider equation (7.1) in a canonical system containing a

quotient:

$$\begin{aligned}
 & u = \dots \\
 & v = \dots \\
 & \text{equations depending on } u, v \\
 (7.1) \quad & w = \frac{u}{v} \\
 & \text{equations depending on } u, v, w.
 \end{aligned}$$

If  $v = 0$  at a point  $t = t_0$ , the standard AD formula for  $n$ -order derivatives of a quotient (1.4) is not applicable, albeit function  $w$  may happen to be holomorphic at this point.

REMARK 14. *If  $w$  is regular at  $t = t_0$  while  $v \rightarrow 0$ , then also  $u \rightarrow 0$  and the expression  $u^{[n]} - \sum_{i=0}^{n-1} r^{[i]} v^{[n-i]} \rightarrow 0$  in formula (1.4). Numerically this effect shows up as the catastrophic subtractive error.*

The convergent Taylor series for  $w$  may be obtained even in this degenerated case due to the enhanced quotient formula established in the following

LEMMA 1. *If*

$$(7.2) \quad v|_{t=t_0} = v'|_{t=t_0} = \dots = v^{[p-1]}|_{t=t_0} = 0, \text{ but } v^{[p]}|_{t=t_0} \neq 0$$

*in the equation  $w = \frac{u}{v}$ , and the function  $w$  is holomorphic at this point, then*

$$(7.3) \quad w^{[n]} = \frac{1}{v^{[p]}} \left( u^{[n+p]} - \sum_{i=0}^{n-1} w^{[i]} v^{[n+p-i]} \right).$$

PROOF. Apply the Leibniz formula of order  $n + p$  to the equation  $u = wv$

$$u^{[n+p]} = w^{[n+p]}v + \dots + w^{[n+1]}v^{[p-1]} + w^{[n]}v^{[p]} + \dots + wv^{[n+p]}$$

observing that at  $t = t_0$  the first  $p$  monomials disappear:

$$u^{[n+p]} = w^{[n]}v^{[p]} + w^{[n-1]}v^{[p+1]} + \dots + wv^{[n+p]}.$$

Obtain the target formula (7.3) by resolving the previous equation in  $w^{[n]}$ .  $\square$

For  $p = 0$  (meaning  $v|_{t=t_0} \neq 0$ ) this formula reduces to the standard formula (1.4) for  $n$ -order derivatives of a quotient. According to formula (7.3), if the conditions (7.2) take place for the equation (7.1) with  $p > 0$ , the *explicitness* of recursive  $n$ -order differentiation breaks:  $u^{[n+p]}$  and  $v^{[n+p]}$  can not be obtained by recursive differentiation of explicit system (7.1) (unless the equations above (7.1) are uncoupled, enabling their recursive differentiation disregarding the rest of the system).

When a canonical system, or generally an algebraic-differential system, gets implicit, the recursive evaluation of the Taylor coefficients requires a sophisticated analysis and algorithms developed in works of [Nedialkov, Pryce 2005]. Here on the contrary we follow a straightforward approach to implicit evaluation of the Taylor coefficients, based on possibility of conversion of an explicit first order system into an implicit (perhaps singular)  $n$ -order ODE. We are interested only in theoretical outcomes of this evaluation, disregarding possible impracticality of the system-to-one-ODE conversion.

## 2. Evaluation of the Taylor coefficients via an implicit singular ODE

Recall that a system of explicit first order ODEs (say in  $x, y, z, \dots$ ) may be transformed (in many ways) to one implicit polynomial ODE in any of the variables, say in  $x$  :

$$(7.4) \quad P(t, x, x', \dots, x^{(n)}) = 0$$

with the polynomial

$$P(T, X_0, X_1, \dots, X_n).$$

We are considering an IVP for this ODE  $x^{(k)}|_{t=0} = a_k, k = 0, 1, \dots, n$  near a point  $(0, a_0, a_1, \dots, a_n)$  satisfying this ODE and such that

$$(7.5) \quad \left. \frac{\partial P}{\partial X_n} \right|_{t=0} = 0$$

In other words,  $(0, a_0, a_1, \dots, a_n)$  is a point of singularity of the ODE, where the condition of existence and uniqueness of the solution is violated. In fact there do exist examples when the solution of (7.4) is not unique.

EXAMPLE 5. *An IVP for the ODE*

$$P = tx' - x = 0$$

at  $t = 0$  has infinitely many solutions  $x(t) = Ct$ .

EXAMPLE 6. *An IVP for the ODE*

$$P = tx' - tx - x = 0$$

at  $t = 0$  has infinitely many solutions  $x(t) = Cte^t$ .

We are to show that the Taylor coefficients of the solution of a polynomial ODE may be evaluated even in a point of singularity of the ODE.

Consider the following process of differentiating ODE (7.4)

$$\frac{d^N}{dt^N} P(t, x, x', \dots, x^{(n)}) = P_N(t, x, x', \dots, x^{(n+N)}); \quad P_0 = P$$

obtaining an infinite sequence

$$(7.6) \quad \begin{aligned} P_1(t, x, \dots, x^{(n+1)}) &= \frac{\partial P(t, x, \dots, x^{(n)})}{\partial X_n} x^{(n+1)} + Q_1(t, x, x', \dots, x^{(n)}) = 0 \\ P_2(t, x, \dots, x^{(n+2)}) &= \frac{\partial P(t, x, \dots, x^{(n)})}{\partial X_n} x^{(n+2)} + Q_2(t, x, x', \dots, x^{(n+1)}) = 0 \\ &\dots \\ P_N(t, x, \dots, x^{(n+N)}) &= \frac{\partial P(t, x, \dots, x^{(n)})}{\partial X_n} x^{(n+N)} + Q_N(t, x, x', \dots, x^{(n+N-1)}) = 0 \\ &\dots \end{aligned}$$

Observe that the coefficient  $\frac{\partial P(t, x, \dots, x^{(n)})}{\partial X_n}$  at the leading derivative in every equation is the same. In this sequence of finite difference equations, values  $x^{(k)}|_{t=0} = a_k, k = 0, 1, \dots, n$  are known, while values  $x^{(n+1)}, x^{(n+2)}, \dots$  are unknowns. (If

not the condition of singularity (7.5), all values  $x^{(n+1)}, x^{(n+2)}, \dots$  could be obtained as the *unique* solutions of the corresponding linear equations). Because of the singularity at  $t = 0$ , the system for unknowns  $x^{(n+1)}, x^{(n+2)}, \dots$  takes the form

$$(7.7) \quad \begin{aligned} Q_2(0, a_0, a_1, \dots, a_n, x^{(n+1)}) &= 0 \\ Q_3(0, a_0, a_1, \dots, a_n, x^{(n+1)}, x^{(n+2)}) &= 0 \\ \dots & \\ Q_N(0, a_0, a_1, \dots, a_n, x^{(n+1)}, \dots, x^{(n+N-1)}) &= 0 \\ \dots & \end{aligned}$$

In principle this polynomial system may have no solutions, a unique solution, or infinitely many of them. We can not always find an expression containing all the solutions. However any of the solutions may be obtained in the following process.

Consider the first equation  $Q_2 = 0$  (in only one unknown  $x^{(n+1)}$ ). If it degenerated into a zero polynomial, assume  $x^{(n+1)} = C_{n+1}$  - an arbitrary constant<sup>1</sup>.

If it degenerated into a *nonzero* polynomial of a *degree zero*, i.e. into a constant  $C_{n+1}(0, a_0, a_1, \dots, a_n)$  which is not zero, system (7.7) has no solutions.

Otherwise  $Q_2$  is at least a linear polynomial in  $x^{(n+1)}$ , having at least one root: generally more than one, say  $m_2 \geq 1$  roots. Choose any of them (or otherwise an arbitrary value  $C_{n+1}$ ) and denote it  $b_{n+1}$ .

Proceed to the next equation  $Q_3 = 0$ , in which again,  $x^{(n+2)}$  is the only unknown, as in the previous one.

If it degenerated into a zero polynomial, assume  $x^{(n+2)} = C_{n+2}$  - an arbitrary constant<sup>1</sup>.

If it degenerated into a *nonzero* polynomial of a *degree zero*, i.e. into a constant  $C_{n+2}(0, a_0, a_1, \dots, a_n, C_{n+1})$ , set it to zero introducing a polynomial relation  $C_{n+2}(0, a_0, a_1, \dots, a_n, C_{n+1}) = 0$ , meaning that the previous constant(s)  $C_{n+1}$  must satisfy this relation (instead of being arbitrary). If it is impossible to satisfy this relation, the previously made choices of the roots or arbitrary constants can not comprise the solution of system (7.7).

Otherwise this polynomial equations is at least linear and has  $m_3 \geq 1$  roots. Choose any of them (or otherwise an arbitrary value  $C_{n+2}$ ) and denote it  $b_{n+2}$ , and so on.

This way we will obtain a sequence  $a_0, a_1, \dots, a_n, b_{n+1}, b_{n+2}, \dots$ , in which some  $b_{n+N}$  are arbitrarily picked values, while others are one of several roots of the respective polynomials. (Indeed, earlier obtained or chosen values  $b_{n+1}, b_{n+2}, \dots$  affect the properties of all the subsequent equations  $Q_N = 0$ , making the set of all the solutions quite sophisticated).

**CONCLUSION 1.** *The finite-difference system (7.7) has a unique solution if all equations (7.7) are higher than zero degree polynomials each being:*

- either linear in the leading unknown,
- or non-linear, but having only one multiple root.

Does every (of many possible) solutions of this finite-difference system generate an analytical element satisfying the source ODE (7.4)? The positive answer is given by the following

<sup>1</sup>In examples 3, 4, one of equations (7.7) does degenerate into a zero polynomial.

Generally on the following steps this constant may be required to satisfy a relation (instead of being arbitrary).

LEMMA 2. *If a solution of the finite difference system (7.7) generates a Taylor expansion (an analytic element) having a nonzero convergence radius, this analytic element is indeed the solutions of the singular IVP (7.4).*

PROOF. Assume we obtained a finite difference solution  $b_{n+1}, b_{n+2}, \dots$  of system (7.7) so that  $a_0, a_1, \dots, a_n, b_{n+1}, b_{n+2}, \dots$  are values of the derivatives defining a holomorphic solution  $y(t)$ . Substitute  $y(t)$  into the sequence of equations (7.6) not expecting that they are necessarily satisfied by  $y(t)$  so that  $P(t, y, y', \dots, y^{(n)}) = \varepsilon(t)$ . Observe that all  $\varepsilon^{(N)}(0) = P_N(t, y, y', \dots, y^{(n+N)})|_{t=0} = 0, N = 0, 1, 2, \dots$  due to the method of obtaining the values  $b_{n+1}, b_{n+2}, \dots$ . Therefore  $\varepsilon(t) \equiv 0$  so that  $y(t)$  is in fact a solution of IVP (7.4).  $\square$

EXAMPLE 7. *At  $t = 0$  all equations (7.7) for the function*

$$x(t) = \frac{e^t - 1}{t}, \quad x(0) = 1$$

*satisfying the singular IVP for the ODE*

$$P = tx' - tx + x - 1 = 0$$

*are linear in the leading unknown:*

$$\begin{aligned} \frac{d^N P}{dt^N} \Big|_{t=0} &= \left[ tx^{(N+1)} + Nx^{(N)} - tx^{(N)} - Nx^{(N-1)} + x^{(N)} \right]_{t=0} = Q_N = \\ &= (N+1)x^{(N)} - Nx^{(N-1)} = 0; \\ x^{(N)} \Big|_{t=0} &= \frac{1}{N+1} \end{aligned}$$

*so that  $x(t)$  is a unique solution of this singular IVP.*

EXAMPLE 8. *The same is true for the IVP*

$$\begin{aligned} P &= tx'' - x = 0; \quad x(0) = 0, \quad x'(0) = 1; \\ \frac{d^N P}{dt^N} \Big|_{t=0} &= \left[ tx^{(N+2)} + Nx^{(N+1)} - x^{(N)} \right]_{t=0} = Q_{N+1} = \\ &= Nx^{(N+1)} - x^{(N)} = 0; \\ x^{(N)} \Big|_{t=0} &= \frac{1}{(N-1)!}, \quad N \geq 1; \end{aligned}$$

CONCLUSION 2. *Switching from explicit equations to the implicit ones in recursive evaluation of  $n$ -order derivatives is what makes the difference between Taylor expansions at points of elementariness vs. Taylor expansions at points-suspects of being non-elementary.*

CONCLUSION 3. *A Taylor expansion at any point of elementariness may be generated by the explicit canonical equations and the formulas of differentiation (1.3-1.5). An expansion generated by an algorithm not belonging to this class is therefore a suspect for representing a function non-elementary at the point. In order to establish non-elementariness at the isolated point, it must be proved that the sequence can not be generated by any system of canonical equations. However no such example is yet known. (The Gamma function and integral are non-elementary in all point of their domain of existence).*





## CHAPTER 8

### Open questions

Elementary functions as a sub-class of holomorphic functions give some new insights into the holomorphic functions in general. On the one hand, elementary functions outline a sub-class of holomorphic functions which are constructively computable (via integration of ODEs as a tool of analytic continuation).

On the other hand elementary functions introduce a finer distinction into types of special points: special points among the holomorphic points. We have learned that in the Gamma function all regular points are non-elementary.

As to the special point  $t = 0$  in functions like  $\frac{\sin t}{t}$ ,  $\frac{e^t-1}{t}$  and similar, we can not yet make such claim. We only know that they can not satisfy any *regular*  $n$ -order polynomial ODE [Gofen 2008, Flanders 2007] - not enough to claim that they are non-elementary at  $t = 0$ .

However, what are the reasons for accepting exactly Definitions 1, 2 of elementariness based on first order explicit systems of ODEs? Why not to consider the following two modified definitions.

DEFINITION 3. (*Stand alone weak elementariness*) Function  $u(t)$  is called stand alone weak elementary if it satisfies some implicit polynomial ODE

$$P(t, u, u', \dots, u^{(n)}) = 0$$

(no matter regular or singular at a particular point).

Due to possibility of system-to-one-ODE conversion (Theorem 4, section 5.2), this class of weak elementary functions widens Definitions 1, 2. In the Gamma function is non-elementary, however the functions  $\frac{\sin t}{t}$ ,  $\frac{e^t-1}{t}$  and similar are unconditionally weak-elementary everywhere: no distinction is made for the point  $t = 0$  despite the fact that integration into it is impossible.

DEFINITION 4. (*Stand alone elementariness*) Function  $u(t)$  is called stand alone elementary at the point  $t = t_0$  if it satisfies some implicit polynomial ODE

$$P(t, u, u', \dots, u^{(n)}) = 0$$

or explicit rational ODE both regular at  $t = t_0$ .

This definition is formally more narrow than Definitions 1, 2, because a regular  $n$ -order ODE trivially converts to the first order system, but not vice versa. If the Definitions 1, 2 were just abandoned in favor of the stand alone elementariness, the Main Theorems for example could not be reproved by the available methods.

That is why Definitions 1, 2 seem to be the most proper in the moment. If the Conjecture [Gofen 2008] is true, the equivalency between an  $n$ -order polynomial ODE and a system of first order explicit ODEs does take place (in the sense explained in the chapter Definitions), hence the stand-alone-elementariness becomes

equivalent to the Definitions 1, 2. This fact too underlines the importance of solving the Conjecture.

Finally, here is a proof for a particular case of the Conjecture for  $m = 2$ . Here the Conjecture is formulated for the source system converted into the form of squares only - without violation of generality due to the Fundamental Transforms.

CONJECTURE 1. *For every component (say  $u_1$ ) of the IVP for polynomial system in squares only*

$$\begin{cases} u'_k = \sum_{i=1}^m a_{ki}u_i^2, & u_k|_{t=t_0} = b_k, \quad k = 1, \dots, m \end{cases}$$

*there exists a regular IVP for  $n$ -order rational ODE*

$$u^{(n)} = \frac{P(t, u, u', \dots, u^{(n-1)})}{Q(t, u, u', \dots, u^{(n-1)}), \quad Q(t, u, u', \dots, u^{(n-1)})|_{t=t_0} \neq 0$$

*having  $u_1$  as a unique solution.*

PROOF. (For  $m = 2$  only). Consider a system

$$\begin{aligned} x' &= a_1x^2 + b_1y^2 \\ y' &= a_2x^2 + b_2y^2. \end{aligned}$$

If  $b_1 = 0$ , the target ODE is  $x' = a_1x^2$ . Now assume  $b_1 \neq 0$ .

$$\begin{aligned} x' &= a_1x^2 + b_1y^2, & y^2 &= (x' - a_1x^2)/b_1 \\ y' &= a_2x^2 + b_2y^2, & y' &= a_2x^2 + \frac{b_2}{b_1}(x' - a_1x^2) \end{aligned}$$

Looking at  $y'$ , observe that all derivatives  $y^{(n)}$  depend on  $x$  and its derivatives only, and they may be expressed as polynomials  $G_n$ :  $y^{(n)} = G_n(x, x', \dots, x^{(n)})$ ,  $n = 1, 2, \dots$ . Utilizing this, differentiate the first equation:

$$\begin{aligned} x^{(n+1)} &= a_1(x^2)^{(n)} + b_1(y^2)^{(n)} = \\ &= a_1(x^2)^{(n)} + 2b_1(yy^{(n)} + ny'y^{(n-1)} + \dots) \\ &= a_1(x^2)^{(n)} + 2b_1(yG_n + nG_1G_{n-1} + \dots), \quad n = 1, 2, \dots \end{aligned}$$

Now observe that  $y$  occurs only in one monomial: that with factor  $G_n(x, x', \dots, x^{(n)})$ . If at least one  $G_n|_{t=t_0} \neq 0$ ,  $y$  may be eliminated and substituted in the following equations, so that we can obtain infinitely many rational ODE's in  $x$  regular at  $t = t_0$ . Otherwise, if all  $G_n|_{t=t_0} = y^{(n)}|_{t=t_0} = 0$ ,  $n = 1, 2, \dots$ , then  $y$  must be a constant so that the first ODE depends only on  $x$ . That concludes the proof.  $\square$

Unfortunately, for  $m > 2$  this method of proof does not work, nor does it work for  $m = 2$  with addition of mixed or linear monomials in the ODEs.

ACKNOWLEDGEMENT 1. *It was a long process to collect the pieces and to recognize that they in fact comprise a coherent picture unifying ODEs and AD. Tracing it all the way back, I owe a deep gratitude to the people who instilled in me the necessary knowledge and mind set, without which this work would never succeed.*

*To begin with, I thank my dearest mathematic teachers - sisters Tatiana and Eleonora Storozhenko in the 60s (the high school No. 116 and Odessa University, Ukraine). It was then (also prompted by discussions with my fellow student Arkadiy Alt) that I first stumbled onto the two crucial questions. Why is the Faa di-Bruno*

*formula so much more complex than other formulas of  $n$ -order differentiation? And what is the particularity at  $t = 0$  of the remarkable limit and this entire function  $\frac{\sin t}{t}$ ?*

*I am also grateful to my colleagues in Hydro-Meteorological Center in the 80s, (Moscow, Russia). There Vladimir Gordin was first to draw my attention to the optimized Taylor method, while Dmitry Sonechkin gave me the papers of Charnyi and Kerner on those "unbelievable" transforms of ODEs.*

*Finally I express many thanks to Christer Kiselman for encouragement, and to George Corliss for helpful advices he made when I first showed him a draft of the Unifying view in 2004, Chicago.*



## CHAPTER 9

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